Bending of Beams or What Does Calculus Have to Do With the Design of Structures?

Introduction

OBJECTIVE: To see how engineers use calculus to design structures.

Structural engineers need to calculate how beams bend, and they do so by using principles of structural mechanics and calculus. In this module, you will investigate some of the ways that engineers use calculus to ensure that the structures they design are both safe and functional.

Before you begin this module, we recommend that you refer to "Maximums, Minimums, and Inflection Points," a JAVA applet included in this supplement. This applet allows you to explore the relationship between the shape of the graph of a function and the values of its first and second derivatives.

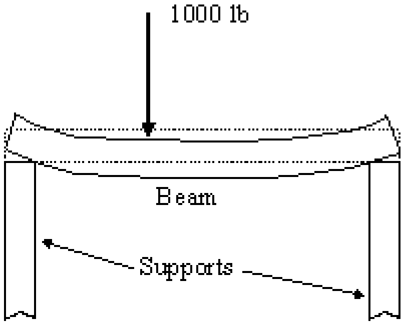
Technology Guidelines

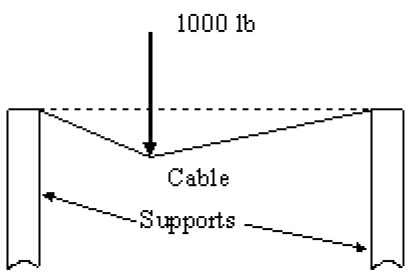
NOTE: If you have just finished a module, restart *Mathematica* or close the *Kernel* before executing a new module.
TO OPEN CELLS, put your cursor on the right cell bracket and double click.
INITIALIZATION CELLS
When asked if you want to ". . . automatically evaluate all the initialization cells in the notebook . . . ," respond by pressing the "Yes" button.
TO STOP AN EXECUTION
Select the *Kernel* pull-down menu and click on *Abort Evaluation.*
ORDER OF EXECUTION
Execute cells in the order given. Do not skip any Input cells within a given notebook.
SAVING NOTEBOOKS
You can save anytime to any directory you choose, and it is wise to save often.
However, before you do your final save, delete all your output by selecting the

Delete All Output selection under the *Kernel* pull-down menu.
EXPERIENCING MAJOR PROBLEMS
Save if appropriate, then shut down *Mathematica* and start it up again.

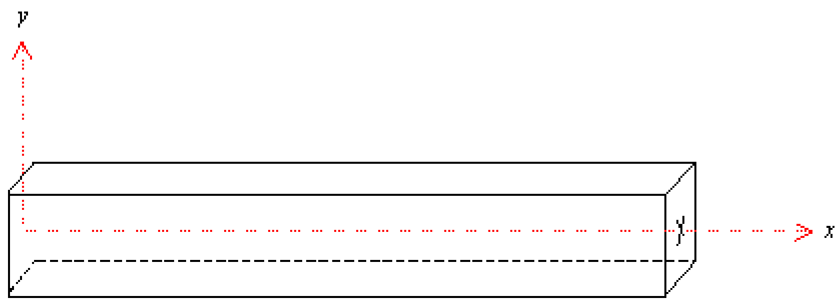
Background Information

A beam is a component of a structure that supports loads by bending. Loads are the forces that act on a structure or structural component due to the weight of objects resting on it, due to its own weight, and due to winds and earthquakes. In contrast to beams, cables and hangars support loads by stretching, columns and arches by compressing, and shafts and rods by twisting. The two figures that follow show how a beam supports a 1000-pound load by bending, and how a cable supports the same load by stretching.





Before any loads are applied to a beam, the long axis through the center of the beam is taken to be straight and horizontal. It is shown as the x-axis in the figure below. When the loads are applied, the beam bends, resulting in a vertical deflection of the long axis. The deflected axis is called the elastic curve, and it is denoted by $y = u(x)$. The independent variable x measures the position along the axis of the unloaded/undeflected beam, and $u(x)$ measures the vertical deflection of the point at position x along the beam. Knowing the deflections, $u(x)$, the structural engineer can calculate the maximum deflections of the beam to ensure that they meet the requirements and specifications for the design. For example, deflections that exceed the maximum allowable deflection may result in damage to windows, wall partitions, ceiling panels, and so on. Therefore, design specifications usually place limits on the maximum deflections.



For a large class of beams, those that are statically determinate, the principles of equilibrium of forces allow the direct calculation of $u''(x)$, the second derivative of the elastic curve, as a function of x. For statically indeterminate beams, we have to start with $u^{(4)}(x)$, the fourth derivative of the elastic curve. In addition, the support conditions can usually be specified. For example, the beam in the first figure above rests on two columns, which prevent the ends of the beam from displacing up or down. Mathematically, these conditions are $u(0) = 0$ and $u(L) = 0$, where L is the length of the beam between the supports. Knowing $u''(x)$, the engineer can integrate this function twice and apply appropriate support conditions to determine $u(x)$, the equation of the elastic curve. If we start with $u^{(4)}(x)$, we need to integrate four times and apply the support conditions, and we will need four conditions instead of two.

In mathematics, the support conditions are often called boundary conditions or initial conditions. The combination of a differential equation and boundary conditions is called a boundary value problem; the combination of a differential equation and initial conditions is called an initial value problem.

Structural engineers are interested in the concavity of a beam's elastic curve. Wherever the elastic curve is concave up, the beam is said to be in positive bending and, wherever it is concave down, the beam is in negative bending. For positive bending, the material above the long axis of the beam is compressed while the material below the long axis is stretched. The situation is reversed for negative bending.

In reinforced concrete beams, concrete is used to resist compression while the steel reinforcement bars or "rebars" are used to resist stretching or tension. Consequently, the structural engineer must design reinforced concrete beams with the rebar on the bottom for positive bending and on the top for negative bending. In some cases, a beam can have positive bending over part of its length and negative bending over the rest. In such a situation, the rebar is switched from the top to the bottom or vice versa at the inflection points of the elastic curve.

Part I: A Cantilever Beam

A Description of the Beam and its Supports

A cantilever beam has one end rigidly fixed in a column or wall and the other end free, as shown in the figure above. The second derivative of the elastic curve for the beam shown above is

$$u''(x)=\frac{wL^2}{2EI}\left[2\left(\frac{x}{L}\right)-\left(\frac{x}{L}\right)^2-1\right]$$

where *w* is the load on the beam (in pounds per inch of length or newtons per meter of length), *L* is the length of the beam, *E* is a material constant known as Young's modulus, and *I* is a constant determined by the geometry of the cross section of the beam. For the support at the wall, the initial conditions are $u(0) = 0$ and $u'(0)=0$ (i.e., the beam cannot move up or down, and it cannot rotate at the support). These are called initial conditions because the value of the function and the derivative are both specified at x=0.

The Elastic Curve and Maximum Deflections

Let's determine the equation for the elastic curve and calculate the maximum deflection of the beam.

Since *E* and *I* are reserved symbols in *Mathematica*, we will use *e* and *i* to represent these quantities.

In[5]:=

```
Clear[u, w, L, e, i, slope, momemt,
  shear];
```

$$u''[x\_] = \frac{w L^2}{2 e i}\left(2\left(\frac{x}{L}\right)-\left(\frac{x}{L}\right)^2-1\right)$$

Out[6]=

$$\frac{L^2 w\left(-1+\frac{2x}{L}-\frac{x^2}{L^2}\right)}{2 e i}$$

Integrate $u''(x)$ to find the slope of the elastic curve. (Don't forget to add the constant of integration.)

In[7]:=

$$u'[x\_] = \int u''[x]\, dx + c1$$

Out[7]=

$$c1 + \frac{w\,(L - x)^3}{6\,e\,i}$$

Integrate $u'(x)$ to find $u(x)$, the elastic curve.

In[8]:=

$$u[x\_] = \int u'[x]\, dx + c2$$

Out[8]=

$$c2 - \frac{w\,(L - x)^4}{24\,e\,i} + c1\,x$$

Apply the initial conditions at the support to determine the constant of integration.

In[9]:=

$$ics = \{u[0] == 0,\ u'[0] == 0\}$$

Out[9]=

$$\left\{c2 - \frac{L^4\,w}{24\,e\,i} == 0,\ c1 + \frac{L^3\,w}{6\,e\,i} == 0\right\}$$

In this case, the solution of the initial condition equations is obvious; however, in general we would need to solve a system of more complicated linear equations to determine c1 and c2. We include that step next for completeness of the process.

In[10]:=

```
soln = Flatten[Solve[ics, {c1, c2}]]
```

Out[10]=

$$\left\{c2 \to \frac{L^4\,w}{24\,e\,i},\ c1 \to -\frac{L^3\,w}{6\,e\,i}\right\}$$

"> About *Mathematica*

Replace c1 and c2 in the elastic curve function with the values found in the previous step.

In[11]:=

```
u[x_] = u[x] /. soln
```

Out[11]=

$$\frac{L^4 w}{24 e i} - \frac{w (L - x)^4}{24 e i} - \frac{L^3 w x}{6 e i}$$

"> About *Mathematica*

The maximum deflection will occur at critical points, or at one of the two ends of the beam. To find the critical values, look for places where $u'(x)$, the slope of the elastic curve, is either 0 or is undefined. There are no values of *x* where $u'(x)$ is undefined.

In[12]:=

```
Solve[u'[x] == 0, x]
```

Out[12]=

$$\left\{\{x \to 0\}, \left\{x \to \frac{1}{2}\left(3 L - i \sqrt{3} L\right)\right\}, \left\{x \to \frac{1}{2}\left(3 L + i \sqrt{3} L\right)\right\}\right\}$$

Since $x = 0$ is not in the interior of the domain $0 \le x \le L$ and the other roots are complex, there are no critical values of *x*. Therefore, the maximum deflection will occur at $x = 0$ or at $x = L$.

In[13]:=

```
u[0]
```

Out[13]=

0

In[14]:=

```
u[L]
```

Out[14]=

$$-\frac{L^4 w}{8 e i}$$

The maximum deflection is at the free end of the beam and is $-\frac{w L^4}{8 EI}$.

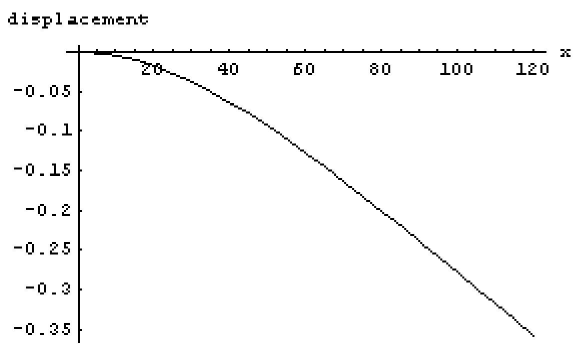
Now let's put in some numbers and plot the elastic curve and its derivative. Let L = 120 in., w = 200 $\frac{\text{lb}}{\text{in}}$, E = 29(10^6) $\frac{\text{lb}}{\text{in}^2}$, and I = 500 in^4.

In[15]:=

```
L = 120; w = 200; e = 29*10^6; i = 500;
```

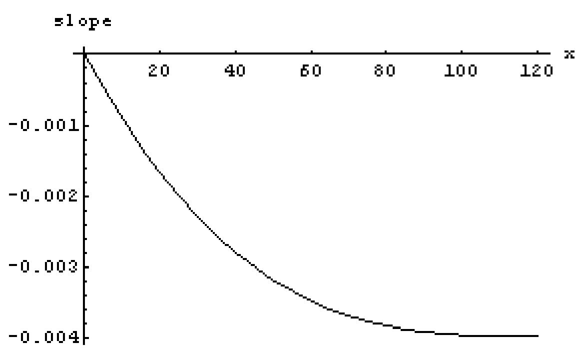
In[16]:=

```
Plot[u[x], {x, 0, L},
 AxesLabel → {"x", "displacement"}];
```



In[17]:=

```
Plot[u'[x], {x, 0, L}, PlotRange → All,
 AxesLabel → {"x", "slope"}];
```



We can also calculate the maximum deflection.

In[18]:=

```
u[L] // N
```

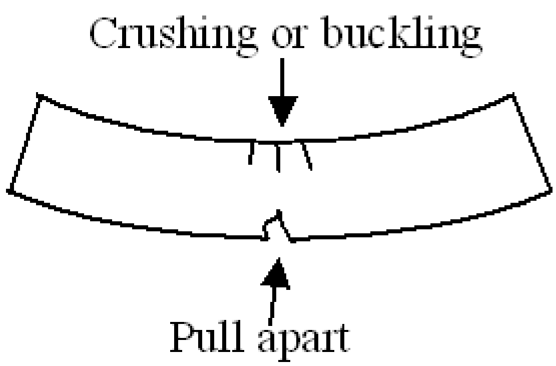
Out[18]=

```
-0.357517
```

The free end of the beam will deflect down 0.358 inches when the specified load is applied.

Will the Beam Break?

The engineer is also interested in how much force is required to break the beam. There are two ways a beam can break. The first is due to excessive bending, and this type of failure is depicted for positive bending in the next figure. To envision negative bending, invert the picture.



The second derivative of the elastic curve, $u''(x)$, is a measure of how much the beam is bent (i.e., its concavity) at any location along its length and can be used to determine where the beam would fail by bending if overloaded. Usually, we are interested in the quantity $EI u''(x)$, which is a force quantity called the bending moment. Bending moments have units of force times distance (e.g., pound-feet). A plot of the bending moment shows that the cantilever beam, with the type of load considered here, would fail at the support if overloaded.

In[19]:=

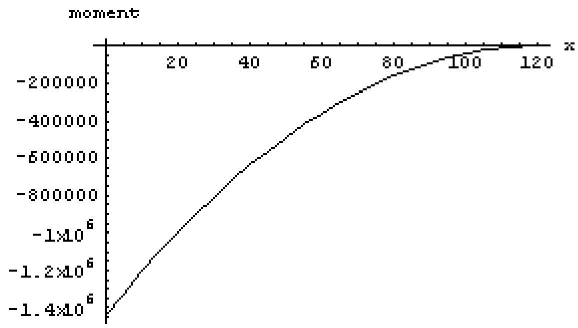
```
moment[x_] = e*i*u''[x] // N // Simplify
```

Out[19]=

$$-100.\ (120. - 1.\ x)^2$$

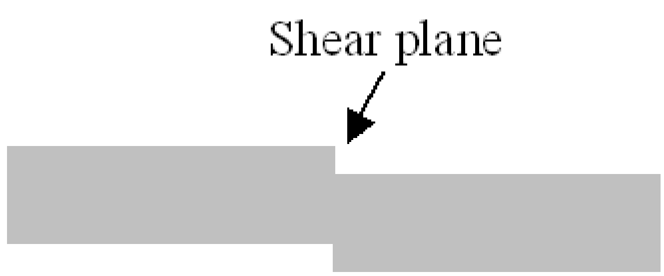
In[20]:=

```
Plot[moment[x], {x, 0, L},
  AxesLabel -> {"x", "moment"}];
```



If a cantilever beam is made of reinforced concrete, steel rebars are placed in the top of the beam along its entire length because u″ (x) is always negative (i.e., negative bending). Consequently, the material in the lower portion of the beam is compressed while the material in the upper portion is in tension. Can you think of a cantilever beam where the load pushes up from underneath causing positive bending?

The second way a beam can fracture is by shearing off, as shown in the next figure. The failure shown in the figure is for positive shear. In cases of negative shear failure, the picture is inverted.



The third derivative of the elastic curve measures the tendency of a beam to shear at any location along its length. Usually we are interested in *EIu'''(x)*, which is called the shear force, and it has units of force (e.g., pounds). A plot of the shear force shows that the cantilever, subject to the load considered here, is most likely to shear off at the support.

In[21]:=

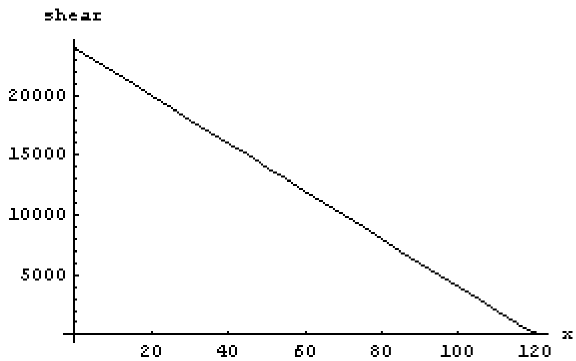
```
shear[x_] = e*i*u'''[x] // N // Simplify
```

Out[21]=

```
24000. - 200. x
```

In[22]:=

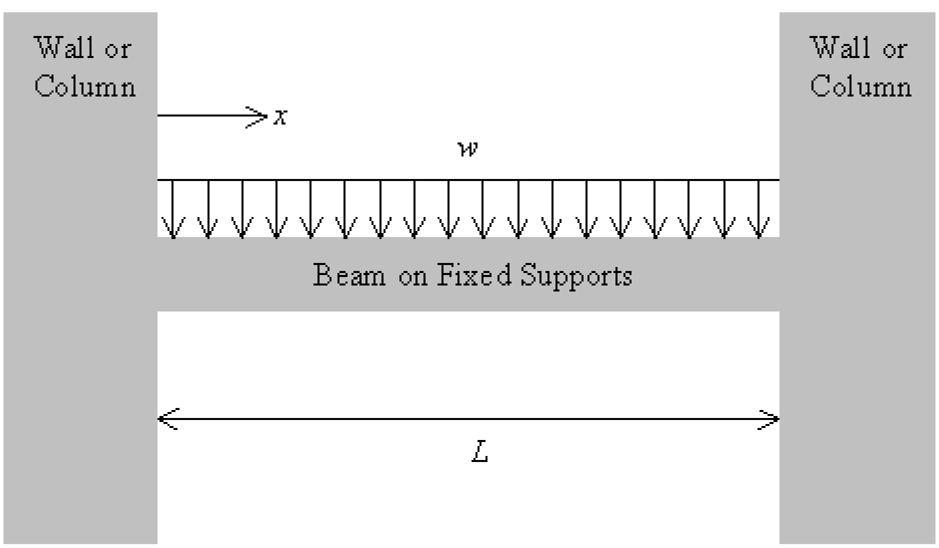
```
Plot[shear[x], {x, 0, L},
  AxesLabel → {"x", "shear"}];
```



Structural engineers must check the shear force and bending moments in beams to ensure the beams are strong enough to support their expected loads.

Part II: A Beam with Fixed Supports

A Description of the Beam and its Supports



The beam shown above has fixed supports at both ends. In this case, we can use the principles of mechanics to determine the fourth derivative of the elastic curve. It is

$u^{(4)}(x) = -\frac{w}{EI}$,

and the boundary conditions are $u(0) = 0$, u(L)=0, $u'(0)=0$, and $u'(L)=0$. The first two conditions specify that the beam cannot be displaced up or down at the two ends, and the second two conditions specify that it cannot rotate at these locations (i.e., the slope of the elastic curve must be 0). Zero displacement and zero rotation characterize a fixed support. These conditions are called boundary conditions rather than initial conditions because they specify conditions on the unknown function u(x) and its derivatives at more than one place in the domain, that is, at x=0 and at x=L.

Now let's perform the following tasks.

Find Elastic Curve, Slopes, Moments, and Shears

1. For generic load *w*, length *L*, material stiffness *E*, and cross-section parameter *I*, determine the functions for the elastic curve, the slope of the elastic curve, the bending moment, and the shear force. To do this, integrate $-\frac{w}{EI}$ four times, adding a new constant of integration each time. Then apply all four boundary conditions to obtain a system of four linear equations in four unknowns (the four constants of integration), and solve the system. *Mathematica* Help will show you how to solve a system of equations using the **Solve[ ]** command.

First, integrate four times, adding a new constant of integration each time.

In[23]:=

```
Clear[u, L, e, i, w, shear, moment, c1,
  c2, c3, c4];
```

```
u''''[x_] = -w / (e * i)
```

Out[24]=

$$-\frac{w}{e\,i}$$

In[25]:=

$$u'''[x\_] = \int u''''[x]\, dx + c1$$

Out[25]=

$$c1 - \frac{w\,x}{e\,i}$$

In[26]:=

```
u''[x_] = ∫u'''[x] dx + c2
```

Out[26]=

$$c2 + c1\,x - \frac{w\,x^2}{2\,e\,i}$$

In[27]:=

```
u'[x_] = ∫u''[x] dx + c3
```

Out[27]=

$$c3 + c2\,x + \frac{c1\,x^2}{2} - \frac{w\,x^3}{6\,e\,i}$$

In[28]:=

```
u[x_] = ∫u'[x] dx + c4
```

Out[28]=

$$c4 + c3\,x + \frac{c2\,x^2}{2} + \frac{c1\,x^3}{6} - \frac{w\,x^4}{24\,e\,i}$$

Apply the boundary conditions, and solve for the constants of integration.

In[29]:=

```
eqns = {u[0] == 0, u[L] == 0, u'[0] == 0,
  u'[L] == 0}
```

Out[29]=

$$\left\{c4 = 0,\ c4 + c3\,L + \frac{c2\,L^2}{2} + \frac{c1\,L^3}{6} - \frac{L^4\,w}{24\,e\,i} = 0,\ c3 = 0,\ c3 + c2\,L + \frac{c1\,L^2}{2} - \frac{L^3\,w}{6\,e\,i} = 0\right\}$$

In[30]:=

```
soln = Flatten[Solve[eqns, {c1, c2, c3, c4}]]
```

Out[30]=

$$\left\{c2 \to -\frac{L^2\,w}{12\,e\,i},\ c1 \to \frac{L\,w}{2\,e\,i},\ c4 \to 0,\ c3 \to 0\right\}$$

Substitute the solutions back into **u[x]** to determine the equation of the elastic curve for the fixed beam, and determine expressions for the slope, bending moment, and shear force.

a) Elastic Curve:

In[31]:=

```
u[x_] = u[x] /. soln // Simplify
```

Out[31]=

$$-\frac{w\,(L-x)^2\,x^2}{24\,e\,i}$$

b) Slope of Elastic Curve:

In[32]:=

```
u'[x] // Simplify
```

Out[32]=

$$-\frac{w\,x\,(L^2-3\,L\,x+2\,x^2)}{12\,e\,i}$$

c) Bending Moment:

In[33]:=

```
moment[x_] = e*i*u''[x] /. soln // Simplify
```

Out[33]=

$$-\frac{1}{12}\,w\,(L^2-6\,L\,x+6\,x^2)$$

d) Shear Force:

In[34]:=

```
shear[x_] = e*i*u'''[x] /. soln // Simplify
```

Out[34]=

$$\frac{1}{2}\,w\,(L-2\,x)$$

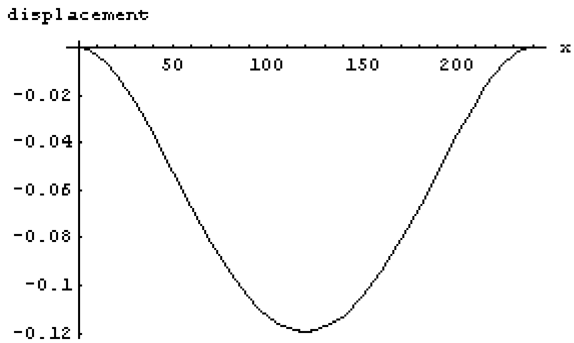
2. Given L = 240 in., w = 200 $\frac{\text{lb}}{\text{in}}$, E = 29($10^6$) $\frac{\text{lb}}{\text{in}^2}$, and I = 500 in^4, plot graphs of the functions found in part (1).

In[35]:=

```
L = 240; w = 200; e = 29*10^6; i = 500;
```

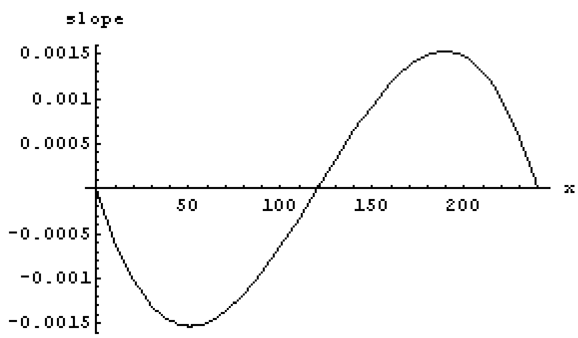
In[36]:=

```
Plot[u[x], {x, 0, L},
 AxesLabel → {"x", "displacement"}];
```



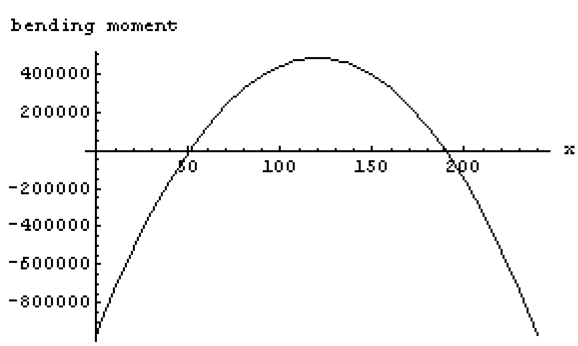
In[37]:=

```
Plot[u'[x], {x, 0, L},
 AxesLabel → {"x", "slope"}];
```



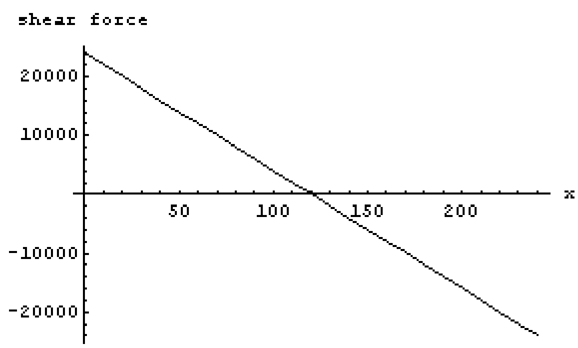
In[38]:=

```
Plot[moment[x], {x, 0, L},
 AxesLabel → {"x", "bending moment"}];
```



In[39]:=

```
Plot[shear[x], {x, 0, L},
 AxesLabel → {"x", "shear force"}];
```



Determine the Extreme Values

3. Determine the absolute maximum and minimum values of the functions found in part (1), and specify where they occur.

From the graph above, it is evident that the maximum and minimum shear forces occur at the two supports. Therefore, the extreme values are as follows:

In[40]:=

```
shear[0]

shear[L]
```

Out[40]=

```
24000
```

Out[41]=

```
-24000
```

Since the shear force is the derivative of the bending moment function, critical values of the bending moment function occur wherever the shear force is 0 or undefined. There are no places where the shear force is undefined.

In[42]:=

```
xcrit = Solve[shear[x] == 0, x]
```

Out[42]=

```
{{x → 120}}
```

Check the values of the bending moment at the critical values and the ends of the beam.

In[43]:=

```
moment[0]

moment[xcrit[[1, 1, 2]]]

moment[L]
```

Out[43]=

```
-960000
```

Out[44]=

```
480000
```

Out[45]=

```
-960000
```

Extreme values of the slope occur where `EIu″` (x), the bending moment, is 0 or undefined, or at the ends of the beam. Note that the bending moment is 0 at the inflection points of the elastic curve where `u″` (x)=0. There are no places where the bending moment function is undefined.

In[46]:=

```
xip = Solve[u″[x] == 0, x] // N
```

Out[46]=

```
{{x → 50.718}, {x → 189.282}}
```

Check the values of the slope at the critical points and the ends of the beam.

In[47]:=

```
u′[0]

u′[xip[[1, 1, 2]]]

u′[xip[[2, 1, 2]]]
```

```
u′[L]
```

Out[47]=

```
0
```

Out[48]=

```
-0.00152898
```

Out[49]=

```
0.00152898
```

Out[50]=

```
0
```

In this case, the maximum and minimum slopes occur at the two inflection points.

The maximum and minimum deflections will occur where the slope is 0 or undefined, or at the ends of the beam. There are no places where the slope is undefined.

In[51]:=

```
xcritdisp = Solve[u′[x] == 0, x]
```

Out[51]=

```
{{x → 0}, {x → 120}, {x → 240}}
```

Check the displacement at the critical value and the ends of the beam.

In[52]:=

```
u[0]

u[120] // N

u[240]
```

Out[52]=

```
0
```

Out[53]=

```
-0.119172
```

Out[54]=

Interpret the Results

4. Specify where the beam would be most likely to fail in bending and where it would be most likely to fail in shear.

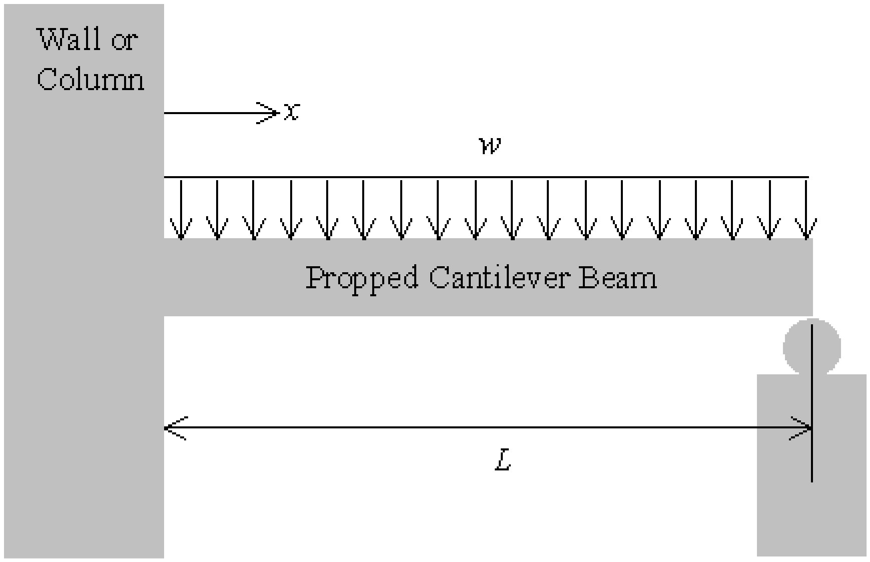
The beam would be most likely to fail in positive bending at the center of the span, and it would most likely fail in negative bending at either of the supports. The shear force is maximum and minimum at the two supports, and so it would likely fail in shear at either one.

5. If the beam were made of reinforced concrete, specify where you would put the tension steel.

The rebars would go in the upper portion of the beam between $x = 0$ and $x = 50.75$ inches and between $x = 189.25$ inches and $x = 240$ inches because, in these portions of the span, the beam is in negative bending. The rebars would go in the lower portion of the beam between $x = 50.75$ inches and $x = 189.25$ inches because this portion of the beam is in positive bending. (Note: The American Concrete Institute's design code requires that the top and bottom steel bars extend a specified distance beyond the inflection points to ensure that the bars are adequately anchored in the concrete so that they can adequately support tension forces.)

You Try It: A Propped Cantilever Beam

A Description of the Beam and its Supports



The beam shown above is a propped cantilever. In this case, we can use the principles of mechanics to determine the fourth derivative of the elastic curve. It is

$u^{(4)}(x)=-\frac{w}{EI}$,

and the boundary conditions are $u(0) = 0$, $u'(0)=0$, $u(L)=0$, and $u''(L)=0$. The first two conditions indicate that the beam cannot move up or down at the left support and it cannot rotate there. The second two conditions indicate that the beam cannot move up or down at the right support, and the bending moment (i.e. $EIy''(x)$) is 0 there.

See if you can accomplish the following tasks.

▪ Find the Elastic Curve, Slopes, Moments, and Shears

1. For generic load *w*, length *L*, material stiffness *E*, and cross-section parameter *I*, determine the functions for the elastic curve, the slope of the elastic curve, the bending moment, and the shear force. Hint: Integrate $-\frac{w}{EI}$ four times, adding a new constant of integration each time. Then apply all four boundary conditions to obtain a system of four equations in four unknowns (the four constants of integration), and solve the system. *Mathematica* Help will show you how to solve a system of equations using the **Solve[ ]** command. To help you, we copied some of the commands from Part II in the next cell. You need to change the items that are in red.

In[55]:=

```
Clear[u, L, e, i, w, shear, moment, c1,
  c2, c3, c4];
```

```
u''''[x_] = -w / (e*i)
```

Out[56]=

$$-\frac{w}{e\,i}$$

In[57]:=

```
u'''[x_] = ∫u''''[x] dx + c1
```

Out[57]=

$$c1-\frac{w\,x}{e\,i}$$

In[58]:=

```
u''[x_] = ∫u'''[x] dx + c2
```

Out[58]=

$$c2+c1\,x-\frac{w\,x^2}{2\,e\,i}$$

In[59]:=

```
u'[x_] = ∫u''[x] dx + c3
```

Out[59]=

$$c3+c2\,x+\frac{c1\,x^2}{2}-\frac{w\,x^3}{6\,e\,i}$$

In[60]:=

```
u[x_] = ∫u'[x] dx + c4
```

Out[60]=

$$c4+c3\,x+\frac{c2\,x^2}{2}+\frac{c1\,x^3}{6}-\frac{w\,x^4}{24\,e\,i}$$

Apply the boundary conditions, and solve for the constants of integration.

In[61]:=

```
bondaryconditions =
  {u[0] == 0, u[L] == 0, u'[0] == 0, u'[L] == 0}
```

Out[61]=

$$\left\{c4 = 0,\ c4 + c3\,L + \frac{c2\,L^2}{2} + \frac{c1\,L^3}{6} - \frac{L^4\,w}{24\,e\,i} = 0,\ c3 = 0,\ c3 + c2\,L + \frac{c1\,L^2}{2} - \frac{L^3\,w}{6\,e\,i} = 0\right\}$$

In[62]:=

```
soln =
  Flatten[Solve[bondaryconditions,
     {c1, c2, c3, c4}]]
```

Out[62]=

$$\left\{c2 \to -\frac{L^2\,w}{12\,e\,i},\ c1 \to \frac{L\,w}{2\,e\,i},\ c4 \to 0,\ c3 \to 0\right\}$$

Substitute the solutions back into **u[x]** to determine the equation of the elastic curve for the fixed beam, and determine expressions for the slope, bending moment, and shear force.

a) Elastic Curve:

In[63]:=

```
u[x_] = u[x] /. soln // Simplify
```

Out[63]=

$$-\frac{w\,(L - x)^2\,x^2}{24\,e\,i}$$

b) Slope of Elastic Curve:

In[64]:=

```
u'[x] // Simplify
```

Out[64]=

$$-\frac{w\,x\,(L^2 - 3\,L\,x + 2\,x^2)}{12\,e\,i}$$

c) Bending Moment:

In[65]:=

```
moment[x_] = e*i*u''[x] /. soln // Simplify
```

Out[65]=

$$-\frac{1}{12} w (L^2 - 6 L x + 6 x^2)$$

d) Shear Force:

In[66]:=

```
shear[x_] = e*i*u'''[x] /. soln // Simplify
```

Out[66]=

$$\frac{1}{2} w (L - 2 x)$$

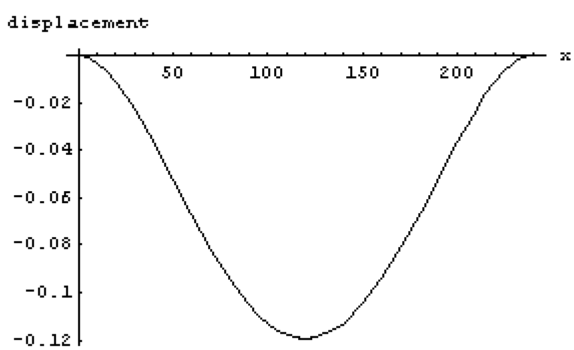
2. Given L = 240 in., w = 200 $\frac{\text{lb}}{\text{in}}$, E = 29($10^6$) $\frac{\text{lb}}{\text{in}^2}$, and I = 500 in^4, plot graphs of the functions found in part (1).

In[67]:=

```
L = 240.; w = 200.; e = 29.*10^6;
i = 500.;
```

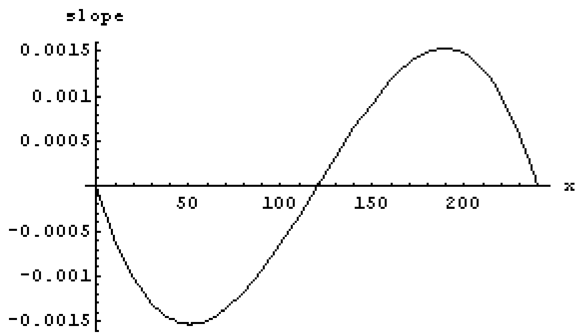
In[68]:=

```
Plot[u[x], {x, 0, L},
  AxesLabel → {"x", "displacement"}];
```



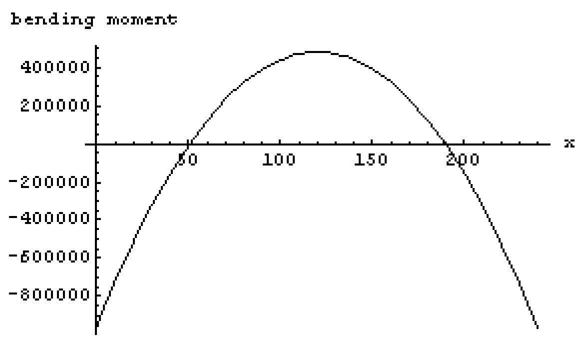
In[69]:=

```
Plot[u'[x], {x, 0, L},
  AxesLabel → {"x", "slope"}];
```



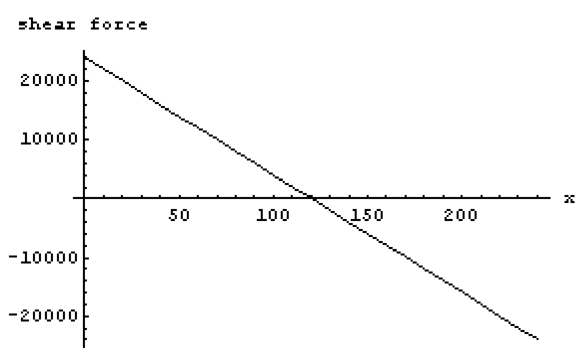
In[70]:=

```
Plot[moment[x], {x, 0, L},
 AxesLabel → {"x", "bending moment"},
 PlotRange → All];
```



In[71]:=

```
Plot[shear[x], {x, 0, L},
 AxesLabel → {"x", "shear force"}];
```



▪ Determine the Extreme Values

3. Determine the absolute maximum and minimum values of the functions found in part (1), and specify where they occur.

From the graph above, it is evident that the maximum and minimum shear forces occur at the two supports. Therefore, the extreme values are as follows:

In[72]:=

```
shear[0]

shear[L]
```

Out[72]=

```
24000.
```

Out[73]=

```
-24000.
```

Since the shear force is the derivative of the bending moment function, critical values of the bending moment function occur wherever the shear force is 0 or undefined. There are no places where the shear force is undefined.

In[74]:=

```
xcrit = Solve[shear[x] == 0, x]
```

Out[74]=

```
{{x → 120.}}
```

Check the values of the bending moment at the critical values and the ends of the beam.

In[75]:=

```
moment[0]

moment[xcrit[[1, 1, 2]]]

moment[L]
```

Out[75]=

```
-960000.
```

Out[76]=

```
480000.
```

Out[77]=

```
-960000.
```

Extreme values of the slope occur where EIu″(x), the bending moment, is 0 or undefined, or at the ends of the beam. Note that the bending moment is 0 at the inflection points of the elastic curve where u″(x)=0. There are no places where the bending moment function is undefined.

In[78]:=

```
xip = Solve[u″[x] == 0, x] // N
```

Out[78]=

```
{{x → 50.718}, {x → 189.282}}
```

Check the values of the slope at the critical points and the ends of the beam.

In[79]:=

```
u′[0]

u′[xip[[1, 1, 2]]]

u′[xip[[2, 1, 2]]]

u′[L]
```

Out[79]=

```
0
```

Out[80]=

```
-0.00152898
```

Out[81]=

```
0.00152898
```

Out[82]=

```
0.
```

The maximum and minimum deflections will occur where the slope is 0 or undefined, or at the ends of the beam. There are no places where the slope is undefined.

In[83]:=

```
xcritdisp = Solve[u'[x] == 0, x]
```

Out[83]=

```
{{x → 0.}, {x → 120.}, {x → 240.}}
```

Check the displacement at the critical value and the ends of the beam.

In[84]:=

```
u[0]

u[xcritdisp[[2, 1, 2]]] // N

u[240]
```

Out[84]=

```
0
```

Out[85]=

```
-0.119172
```

Out[86]=

```
0.
```

▪ Interpret the Results

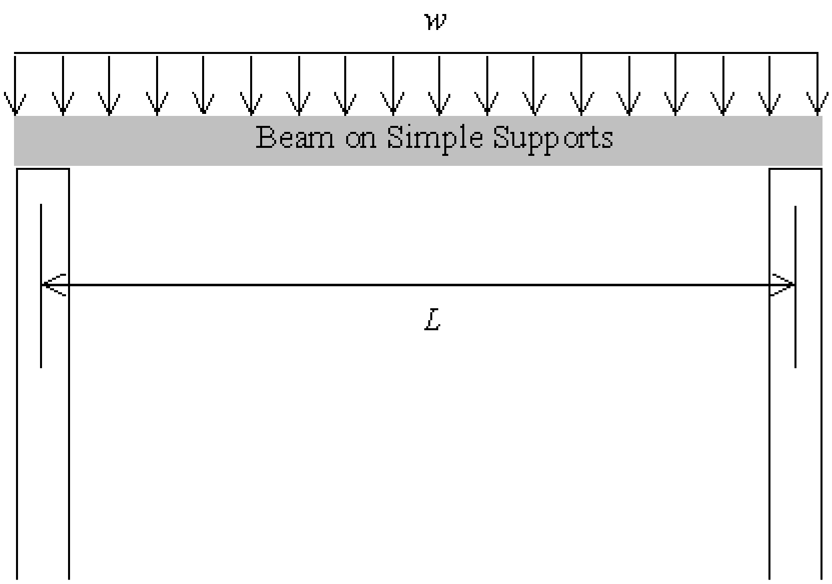
4. Specify where the beam would be most likely to fail in bending and where it would be most likely to fail in shear.

5. If the beam were made of reinforced concrete, specify where you would put the tension steel.

You Try It: A Beam on Simple Supports

▪ A Description of the Beam and its Supports

If a beam simply rests on a support without being rigidly fixed to it, the support is said to be a simple support. A simple support prevents vertical movement of the beam at the support location, but it does not prevent rotation. If the simple support is at the end of a beam, then the bending moment is 0 at that location. The support can be a bearing wall, a column, or a bridge pier. For example, the support at the right end of the propped cantilever beam in the preceding problem is a simple support. The following figure shows a beam on two simple supports.



In this case, the fourth derivative of the elastic curve is $u''''(x)=-\frac{w}{EI}$ and the boundary conditions are $u(0) = 0$, $u''(0)=0$, $u(L)=0$, and $u''(L)=0$.

See if you can accomplish the following tasks.

▪ Find the Elastic Curve, Slopes, Moments, and Shears

1. For generic load *w*, length *L*, material stiffness *E*, and cross section parameter *I*, determine the functions for the elastic curve, the slope of the elastic curve, the bending moment, and the shear force? Hint: Integrate $-\frac{w}{EI}$ four times, adding a new constant of integration each time. Then apply all four boundary conditions to obtain a system of four equations in four unknowns (the four constants of integration), and solve the system. *Mathematica* Help will show you how to solve a system of equations using the **Solve[ ]** command. This time we leave you on your own, but you can copy and paste some of the commands that we used in the preceding sections.

2. Given L = 240 in., w = 200 $\frac{\text{lb}}{\text{in}}$, E = 29($10^6$) $\frac{\text{lb}}{\text{in}^2}$, and I = 500 in^4, plot graphs of the functions found in part (1).

Determine the Extreme Values

3. Determine the absolute maximum and minimum values of the functions found in part (1), and specify where they occur.

Interpret the Results

4. Specify where the beam would be most likely to fail in bending and where it would be most likely to fail in shear.

5. If the beam were made of reinforced concrete, specify where you would put the tension steel.

About *Mathematica*

The **Flatten[ ]** command is used to remove a nested set of curly brackets, **{ }**, that is included inside the list of solution rules generated by the **Solve[ ]** command. In a cell that follows, we will want to use the list of solution rules to substitute back into the displacement, slopes, moment, and shear functions. This won't work unless the inner curly brackets are removed with the **Flatten[ ]** command. To learn more about lists and parts of lists, refer to the "Overview of *Mathematica"* module included with this supplement. To learn more about the **Flatten[ ]** command, pull down the Help menu, select the Help Browser, and type **Flatten**. Go Back.

The command **u[x_]=u[x]/.soln** redefines **u[x]** by replacing the constants of integration in **u[x]** with the values specified in the list of solution rules that was obtained from the **Solve[ ]** command. The new **u[x]** that is formed in this way replaces the constants with the values obtained from the solution of the boundary or initial condition equations. Go Back.
