Visualizing and Interpreting the Divergence Theorem

Introduction

OBJECTIVE: Observe that surface integrals are difficult to evaluate, even using *Mathematica*. See that using parametrizations to evaluate flux surface integrals and applying the Divergence Theorem can help with integral evaluations.

You will see that surface integrals are still difficult to set up and to evaluate, even with *Mathematica*, but parametrizations can assist in evaluating these difficult flux surface integrals. You will also verify the Divergence Theorem, just as you have done with Green's Theorem. This module is an example; however, those familiar with *Mathematica* can adjust the code to solve other problems.

Technology Guidelines

NOTE: If you have just finished a module, restart *Mathematica* or close the *Kernel* before executing a new module.
TO OPEN CELLS, put your cursor on the right cell bracket and double click.
TO STOP AN EXECUTION
Select the *Kernel* pull-down menu and click on *Abort Evaluation.*
ORDER OF EXECUTION
Execute cells in the order given. Do not skip any Input cells within a given notebook.
SAVING NOTEBOOKS
You can save anytime to any directory you choose, and it is wise to save often.
However, before you do your final save, it is a good idea to delete all your output by selecting the
Delete All Output selection under the *Kernel* pull-down menu.
EXPERIENCING MAJOR PROBLEMS
Save if appropriate, then shut down *Mathematica* and start it up again.

Part I: Visualizing the Problem

We select the object formed by a hemisphere of radius 1, topped with a cylinder of radius 1 and height 1, with a circular disk on top on the cylinder. This object is similar to the one to which the method of moments was applied in the previous chapter. The force we choose has components **{-x² -4x y,-6 y z, 12z}**.

In[1]:=

```
Off[General::spell]

Off[General::spell1]

Clear[x, y, z, t, force]

force := {-x^2 -4 x y, -6 y z, 12 z}
```

Before we apply the Divergence Theorem, we plot the figure. We need to load three graphing packages in order to show our plots.

In[5]:=

```
<<Graphics`ParametricPlot3D`

<<Graphics`Graphics3D`
<<Graphics`PlotField3D`
```

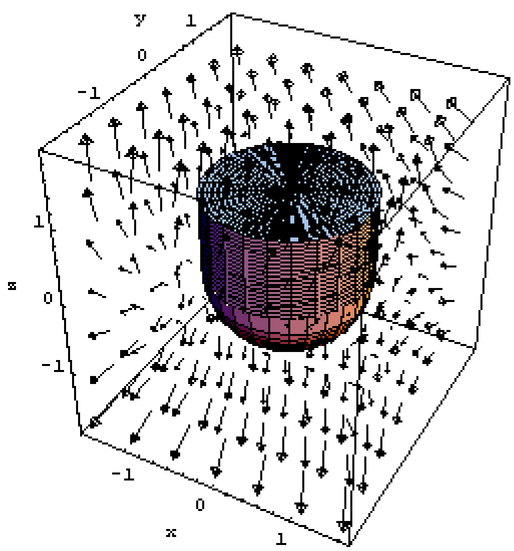
Out[6]=

```
Null^2
```

In[7]:=

```
Clear[x, y, z, r, θ, t]

forceplot = PlotVectorField3D[force, {x, -1.5
  {y, -1.5, 1.5}, {z, -1.5, 1.5}, VectorHeads
  DisplayFunction -> Identity];

hemisplot = CylindricalPlot3D[-√(1 - r^2) , {r, 0
  {θ, 0, 2 π}, AxesLabel -> {x, y, z},
  DisplayFunction -> Identity];

topplot = CylindricalPlot3D[1, {r, 0, 1}, {θ, 0
  AxesLabel -> {x, y, z}, DisplayFunction -> Id
```

```
cylpts = Table[{Cos[t] , Sin[t],
 u}, {t, 0, 2 Pi, Pi / 12},
 {u, 0, 1, .05}];

cylplot = ListSurfacePlot3D[cylpts,
  DisplayFunction -> Identity];

Show[hemisplot, cylplot, topplot, forceplot,
 DisplayFunction :> $DisplayFunction];
```



Part II: Evaluating the Divergence Integral

The divergence integral over the volume is easy to set up and calculate. We begin by first loading a package that can perform certain vector computations, including the divergence, curl, and cross product.

In[14]:=

```
<<Calculus`VectorAnalysis`
```

In[15]:=

```
Clear[x, y, z]

Print["The divergence of this force is ",
 divf = Div[force, Cartesian[x, y, z]]]
```

```
Print["The value of the divergence integral
 totalflux = Integrate[divf, {x, -1, 1}, {y, -
   {z, -√(1 - x² - y²) , 1}]]
```

```
The divergence of this force is
 12 - 2 x - 4 y - 6 z
```

```
The value of the divergence integral is
 37 π
 ────
  2
```

"> About *Mathematica*

Part III: Finding Surface Integrals

The surface integrals evaluating the flux across each surface require considerable set up. As you recall, parametrizations are very convenient in determining line integrals. Likewise, parametrizations can facilitate the computation of surface integrals. We use spherical coordinates to write the equation of the sphere parametrically and follow the procedure outlined in **Section 15.6** of your text.

▪ Hemisphere

In[18]:=

```
Clear[x, y, z, r, θ, ϕ]

x = Sin[θ] Sin[ϕ];

y = Cos[θ] Sin[ϕ];

z = Cos[ϕ];

r = {x, y, z};

rtheta = D[r, θ];

rphi = D[r, ϕ];
```

```
cr = Cross[rtheta, rphi] // Simplify;

Print[
 "The integrand for the hemispherical surfac
 integrand = force.cr // Simplify]

Print["The value of the hemispherical surfac
 hemisflux = Integrate[integrand, {θ, 0, 2 π},
 "or ",
nhemisflux = N[hemisflux]]
```

The integrand for the hemispherical surface integral is

$$12 \cos[\phi]^2 \sin[\phi] - 6 \cos[\theta]^2 \cos[\phi] \sin[\phi]^3 - \sin[\theta]^2 (4 \cos[\theta] + \sin[\theta]) \sin[\phi]^4$$

The value of the hemispherical surface integral is $\frac{19\pi}{2}$ or 29.8451

The equations of the cylinder and the top can be written using cylindrical coordinates, with z and θ the parameters for the cylinder and r = 1.

▪ Cylinder

In[28]:=

```
Clear[x, y, z]

x = Cos[θ];

y = Sin[θ];

z = z;

r = {x, y, z};

rtheta = D[r, θ];
```

```
rz = D[r, z];

cr = Cross[rtheta, rz] // Simplify;

Print[
 "The integrand for the cylindrical surface
 integrand = force.cr // Simplify]

Print["The value of the cylindrical surface
 cylflux = Integrate[integrand, {θ, 0, 2 π}, {z
ncylflux = N[cylflux]]
```

```
The integrand for the
   cylindrical surface integral is
 -Cos[θ]^3 - 4 Cos[θ]^2 Sin[θ] - 6 z Sin[θ]^2
```

```
The value of the cylindrical
   surface integral is -3 π or -9.42478
```

Top

Polar coordinates *r* and θ will be the parameters for the top. Recall that for the top, the normal is always in the increasing *z* direction. Also, the delta area form of the area must contain the polar coordinate *r,* which varies from 0 to 1.

In[38]:=

```
Clear[x, y, z, r]

x = Cos[θ];

y = Sin[θ];

z = 1;

r = {x, y, z};

normal = {0, 0, 1};
```

```
Print[
 "The integrand for the surface integral for
 integrand = force.normal // Simplify]

Print["The value of the surface integral for
 topflux = Integrate[R integrand, {θ, 0, 2 π},
 " or ",
ntopflux = N[topflux]]
```

```
The integrand for the
   surface integral for the top is 12
```

```
The value of the surface integral
   for the top is 12 π or 37.6991
```

Part IV: Verifying the Divergence Theorem

Compare your answers to verify the Divergence Theorem.

In[46]:=

```
Print["The total of the surface integrals is
 totalsurfaceflux = hemisflux + cylflux + topfl
 N[totalsurfaceflux]]

If[totalsurfaceflux <= totalflux + .001 &&
  totalsurfaceflux >= totalflux - .001,
 Print["The Divergence Theorem is validated
 Print["there's a problem"]]
```

The total of the surface integrals is

$\frac{37\pi}{2}$ or 58.1195

```
The Divergence Theorem is validated.
```

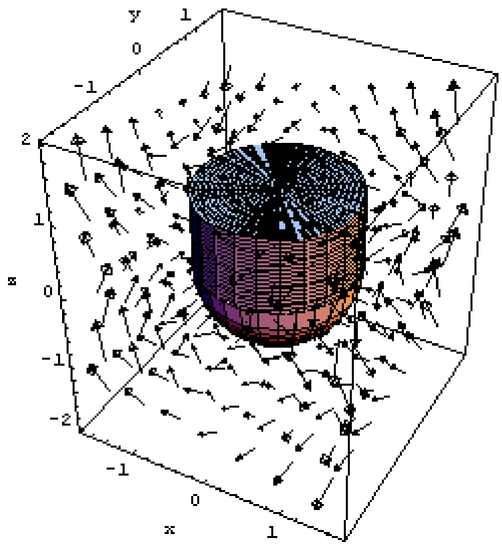
You Try It

Visualize and Evaluate Other Divergence Integrals

You can change the entries in red to define a different force function and then to visualize it.

In[48]:=

```
Clear[x, y, z, r, θ]

force := {Cos[10 z], Sin[10 z], x y}

forceplot = PlotVectorField3D[force, {x, -1.5
  {y, -1.5, 1.5}, {z, -1.5, 1.5}, VectorHeads
  DisplayFunction -> Identity];

hemisplot = CylindricalPlot3D[-√(1 - r^2), {r, 0
  {θ, 0, 2 π}, AxesLabel -> {x, y, z},
  DisplayFunction -> Identity];

topplot = CylindricalPlot3D[1, {r, 0, 1}, {θ, 0
  AxesLabel -> {x, y, z}, DisplayFunction -> Id

cylpts = Table[{Cos[t] , Sin[t],
 u}, {t, 0, 2 Pi, Pi / 12},
 {u, 0, 1, .05}];

cylplot = ListSurfacePlot3D[cylpts,
  DisplayFunction -> Identity];

Show[hemisplot, cylplot, topplot, forceplot,
 DisplayFunction :> $DisplayFunction];
```



Now, see if *Mathematica* can find the divergence integral.

In[56]:=

```
Clear[x, y, z]

Print["The divergence of this force is ",
 divf = Div[force, Cartesian[x, y, z]]]

Print["The value of the divergence integral
 totalflux = Integrate[divf, {x, -1, 1}, {y, -
   {z, -√(1 - x² - y²), 1}]]
```

```
The divergence of this force is 0

The value of the divergence integral is 0
```

▪ Selection of Closed Region to Maximize Flux

Section 16.8, Exercise 24

In this problem, you are to find the dimensions of a particular rectangular box for which the total flux of a force with components *{-* x^2 *-4x y,-6 y z, 12z}* is maximized.

Visualizing the Problem

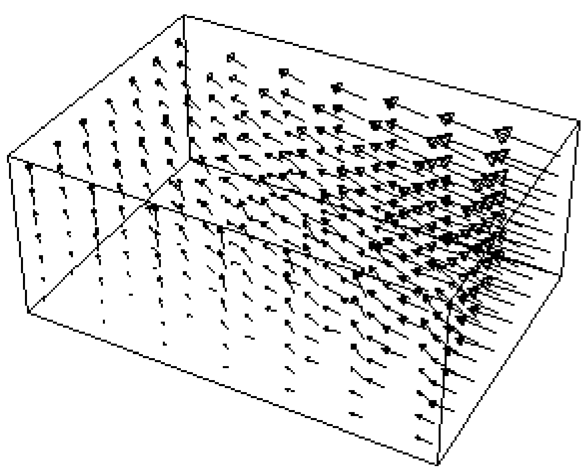
If you have not already loaded the packages used below, be sure to do that. The force here is the same as in Part I, but we are now asked to find the coordinates of a rectangular box that will maximize the outward flux. Before we apply the Divergence Theorem, let's plot the figure.

In[59]:=

```
<< Calculus`VectorAnalysis`

<< Graphics`PlotField3D`
```

In[61]:=

```
Clear[x, y, z, t, force]

force := {-x^2 - 4 x y, -6 y z, 12 z}

forceplot = PlotVectorField3D[force, {x, 0, 3}
  {z, 0, 1}, VectorHeads -> True, ScaleFactor -
  AxesLabel -> {x, y, z}];
```



Evaluating and Maximizing the Divergence Integral

In[64]:=

```
divf = Div[force, Cartesian[x, y, z]];
```

```
divergenceintegral =
 Integrate[divf, {z, 0, 1}, {y, 0, b}, {x, 0, a
```

Out[65]=

$$-a\,b\,(-9+a+2\,b)$$

We will find the values of *a* and *b* that maximize this function by computing the two first partials with respect to *a* and *b* and equating them both to 0.

In[66]:=

```
ma = D[divergenceintegral, a];

mb = D[divergenceintegral, b];

solab = Solve[{ma == 0, mb == 0}, {a, b}]
```

Out[68]=

$$\left\{\{a\to 0,\ b\to 0\},\ \left\{a\to 0,\ b\to \frac{9}{2}\right\},\ \left\{a\to 3,\ b\to \frac{3}{2}\right\},\ \{a\to 9,\ b\to 0\}\right\}$$

Based on these results, what answer will you give?

Evaluate the divergence integral at these values for *a* and *b*.

In[69]:=

```
divergenceintegral /. solab
```

Out[69]=

$$\left\{0,\ 0,\ \frac{27}{2},\ 0\right\}$$

You should check this divergence integral out for other values of a and b and see if the your result is validated. Also, how is this value for the divergence integral related to the value you arrived at in Part II for this same force? Does the fact that that answer is larger contradict your having found the maximum here?

About *Mathematica*

In computing multiple integrals, *Mathematica* starts with the variable and bounds on the extreme right of the **Integrate** command and then continues the evaluation from right to left. Go back.
