First-Order Differential Equations and Slope Fields

Note: You may notice differences between this Maple worksheet and the equivalent Mathematica notebook. These differences were introduced to preserve the content of these modules and were necessary because of major functional differences between Maple and Mathematica.

Introduction

OBJECTIVE: Visualize the slope fields and solution curves for selected first-order differential equations.

This module contains a special command that plots slope fields and selected solution curves for first-order differential equations. You can use it to obtain solutions for related problems in the text. In addition, you can use the slope field command to study a wide variety of first-order differential equations and to analyze the long-term behavior of solutions.

Technology Guidelines

NOTE: If you have just finished a worksheet, **restart** *Maple* before executing a new worksheet.
TO OPEN SECTIONS,
Click on the **PLUS** sign at the left hand side of the screen *or* select **Expand All Sections** from the **View** drop down menu.

TO STOP AN EXECUTION
Click on **STOP** button from the toolbar.

ORDER OF EXECUTION
Execute commands in the order given. Do not skip any *Maple* Input lines within a given worksheet

Alternatively, you can execute the entire worksheet by selecting the **Execute Worksheet** command from the **Edit** drop down menu.

SAVING WORKSHEETS.

You can save anytime to any directory you choose, and it is wise to save often.
EXPERIENCING MAJOR PROBLEMS
Save if appropriate, and then shut down *Maple* and start it up again.

Part I: Drawing Slope fields and Solution Curves

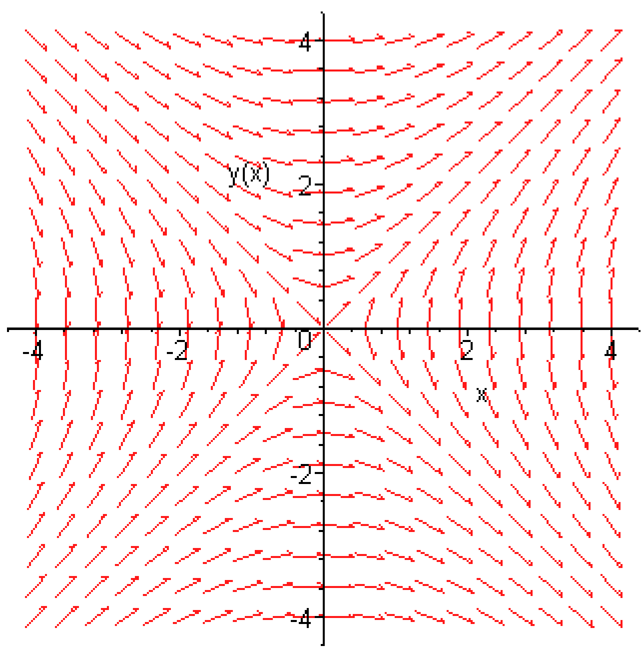
The **DEplot(ode, variable, range1, range2, initialconditions)** command in the next section generates a slope field and draws selected solution curves for first-order differential equations of the form $\frac{dy}{dx} = f(x, y)$. The arguments of the command are **ode**, the differential equation to be plotted, **variable**, the variable the ODE is to be solved for, **range1,** the range of the independant variable, **range2**, the range of the dependant variable, and **initialconditions**, the list of initial conditions. If no initial conditions are listed, only the slope field is plotted. Here's how it works for the differential equation $\frac{dy}{dx} = \frac{x}{y}$. First, we plot only the slope field. ODE stands for ordinary differential equation.

Before beginning, we load the **DEtools** package.

```
> restart:
  with(DEtools):
```

```
> ode := diff(y(x),x)=x/y(x);
  DEplot( ode, y(x), x=-4..4, y(x)=-4..4 );
```

$$ode := \frac{d}{dx}\mathrm{y}(x) = \frac{x}{\mathrm{y}(x)}$$



Include a list of initial conditions to obtain some solution curves. Each of the four initial conditions will usually give a different solution curve.

```
> DEplot( ode, y(x), x=-4..4, y(x)=-4..4,
  [[y(0)=2],[y(1)=1], [y(1)=-1], [y(0)=-2]] );
```

Warning, plot may be incomplete, the following errors(s) were issued:

division by zero

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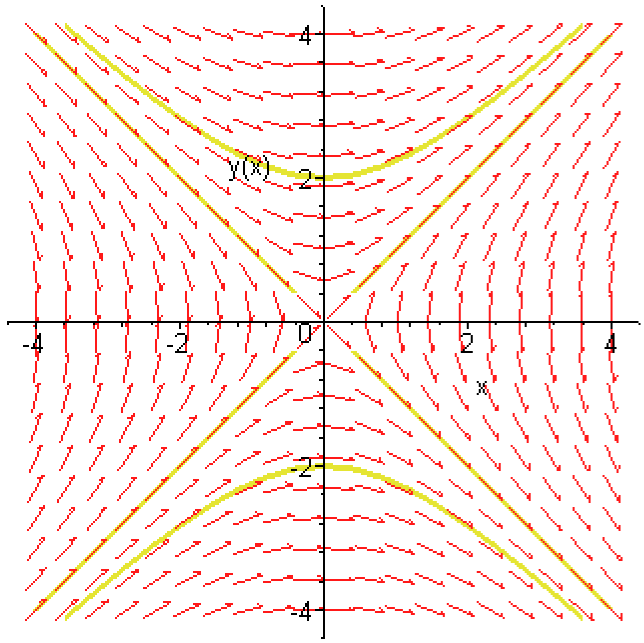
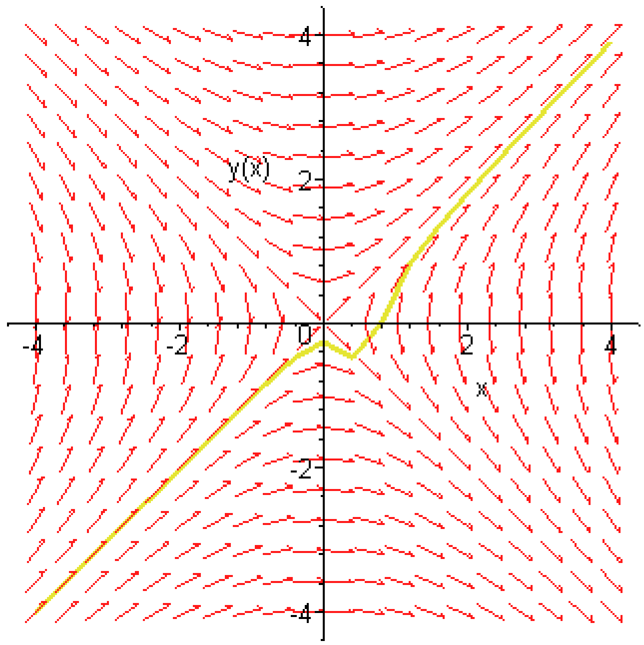
division by zero



You will notice in the graph below that graph looks funny around $x = 0$. When the solution curve passes through the point (1, 1/2) it cannot be expressed as a function where y is a function of x. Here's what happens.

```
> DEplot( ode, y(x), x=-4..4, y(x)=-4..4,
  [[y(1)=1/2]] );
```



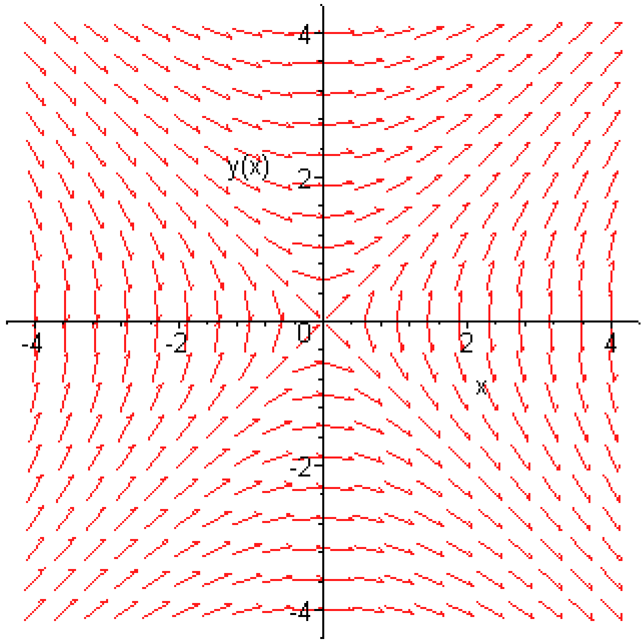
When this happens it is probably better to hand sketch some representative solution curves on the slope field.

You Try It: Part I

It's fun! Make up some of your own differential equations of the form $\frac{dy}{dx} = f\ (x,y)$ and see what kind of patterns you can generate. Can you form a differential equation that will give solution curves that are ellipses? hyperbolas? Just try it by changing the function of *x* and *y* on the right side of the differential equation **ode**.

```
> ode := diff(y(x),x)=x/y(x);
  DEtools[DEplot]( ode, y(x), x=-4..4, y(x)=-4..4 );
```

$$ode := \frac{d}{dx}\,\mathrm{y}(x) = \frac{x}{\mathrm{y}(x)}$$



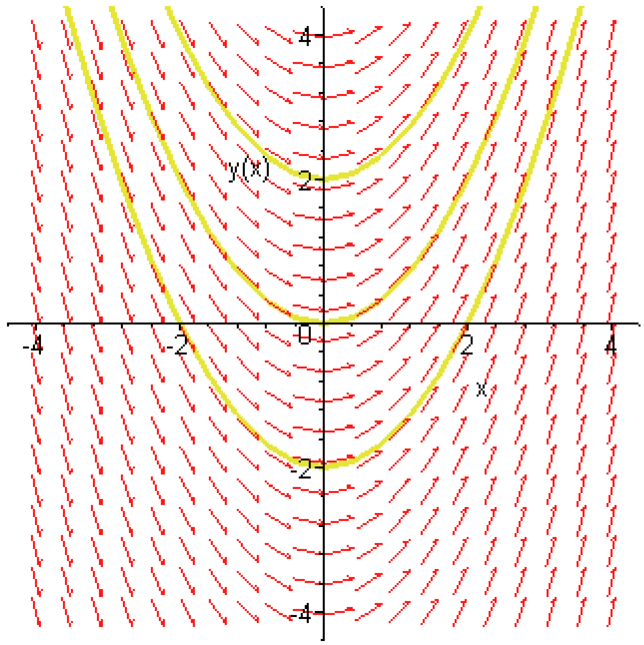
Part II: Antiderivatives

If we consider differential equations of the form $\frac{dy}{dx} = f(x)$, then the solutions for y as a function of x are simply the antiderivatives of $f(x)$; that is, $y = \int f(x)\, dx$. Let's look at some slope fields and solution curves for differential equations of this kind.

First, we consider $\frac{dy}{dx} = x$.

```
> ode := diff(y(x),x)=x;
  DEplot( ode, y(x), x=-4..4, y(x)=-4..4, [[y(0)=-2], [y(0)=0], [y(0)=2]] );
```

$$ode := \frac{d}{dx}\,y(x) = x$$



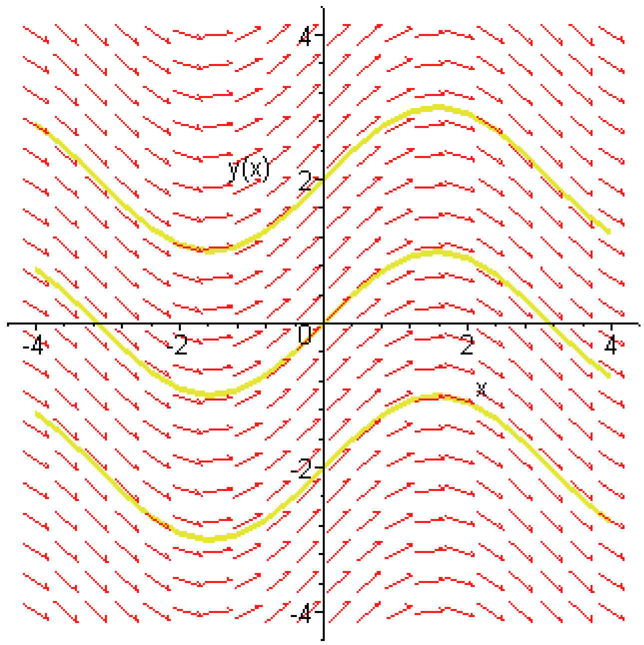
Notice that the slopes in the field above are constant along any vertical line. This shows graphically that the derivative of *y* with respect to *x* is not a function of *y*, as reflected in the differential equation $\frac{dy}{dx} = x$. Along any horizontal line, however, the slopes vary with *x,* and the pattern shown in the slope field above is consistent with the differential equation $\frac{dy}{dx} = x$.

The slopes are negative when *x* is negative, positive when *x* is positive, and 0 when *x* is 0.

Let's try a periodic function, $\frac{dy}{dx} = \cos(x)$.

```
> ode := diff(y(x),x)=cos(x);
  DEplot( ode, y(x), x=-4..4, y(x)=-4..4, [[y(0)=-2], [y(0)=0], [y(0)=2]] );
```

$$ode := \frac{d}{dx} y(x) = \cos(x)$$



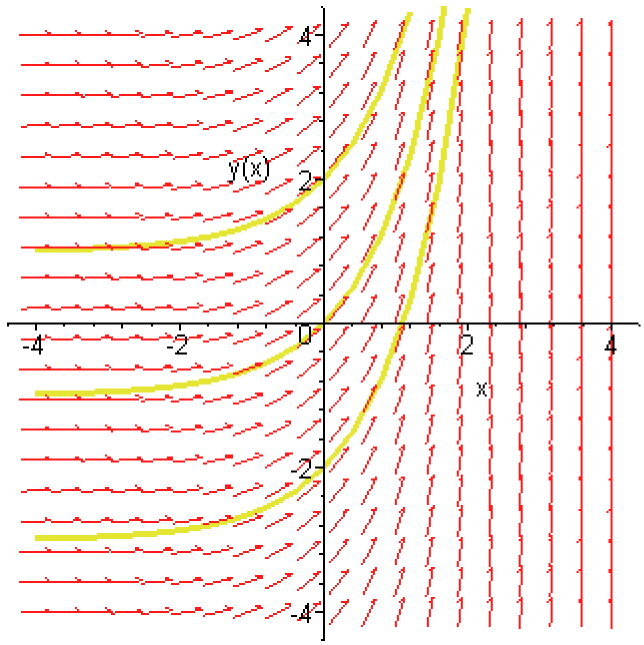
You Try It: Part II

Use the slope field command to study some antiderivative problems of the form $\frac{dy}{dx} = f(x)$

where you pick the function and initial conditions to replace **exp(x)** and the initial conditions (in square brackets) in the next input cell.

```
> ode := diff(y(x),x)=exp(x);
  DEtools[DEplot]( ode, y(x), x=-4..4, y(x)=-4..4, [[y(0)=-2], [y(0)=0], [y(0)=2]] );
```

$$ode := \frac{d}{dx} y(x) = e^x$$

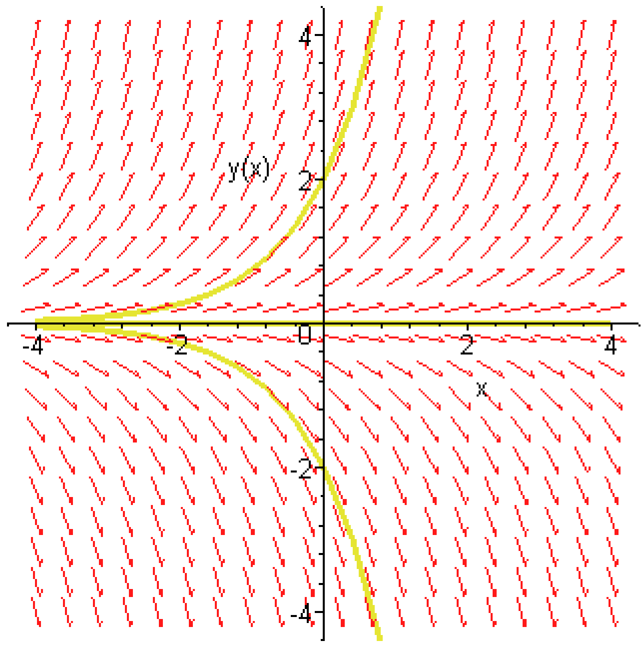


Part III: Autonomous Differential Equations

Differential equations of the form $\frac{dy}{dx} = f(y)$ are called autonomous differential equations. Let's look at the autonomous differential equation $\frac{dy}{dx} = y$. Despite its simplicity, this is surely the most famous of all first-order, autonomous differential equations. Let's draw the slope field and some solution curves.

```
> ode := diff(y(x),x)=y(x);
  DEplot( ode, y(x), x=-4..4, y(x)=-4..4, [[y(0)=-2], [y(0)=0], [y(0)=2]] );
```

$$ode := \frac{d}{dx}\,y(x) = y(x)$$



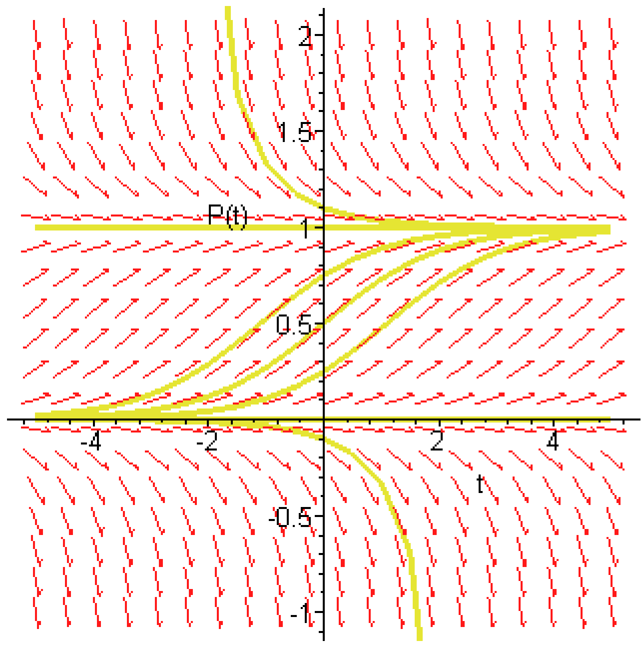
Notice that in the slope fields for $\frac{dy}{dx} = y$, the slopes are constant along any horizontal line and vary linearly along any vertical line. This is the reverse of what we observed for the antiderivative problems of the form $\frac{dy}{dx} = f(x)$.

The logistic equation is another interesting autonomous first-order differential equation. Let's draw a slope field and some solution curves for a specific one, say, $\frac{dP}{dt} = P\ (1\text{-}P)$. For some initial values the solution curves go off to infinity and have vertical asymptotes, however, the solution curves as drawn are okay. Try changing the value for epsilon to see what happens to the solutions with vertical asymptotes.

```
> ode := diff(P(t),t)=P(t)*(1 - P(t));
  epsilon := 1/10;
  DEplot( ode, P(t), t=-5..5, P(t)=-1..2, [[P(0)=-epsilon], [P(0)=0], [P(0)=1/4], [P(0)=1/2], [P(0)=
  [P(0)=1], [P(0)=1+epsilon]] );
```

$$ode := \frac{d}{dt} P(t) = P(t)\,(1 - P(t))$$

$$\varepsilon := \frac{1}{10}$$



In terms of the long-term behavior of the solutions, the slope field graph shown above suggests the following: $\lim_{t \to \infty} P(t) = -\infty$ when $P(0) < 0$; $P(t) = 0$ for all values of t when $P(0) = 0$; and, $\lim_{t \to \infty} P(t) = 1$ when $P(0) > 0$.

You Try It: Part III

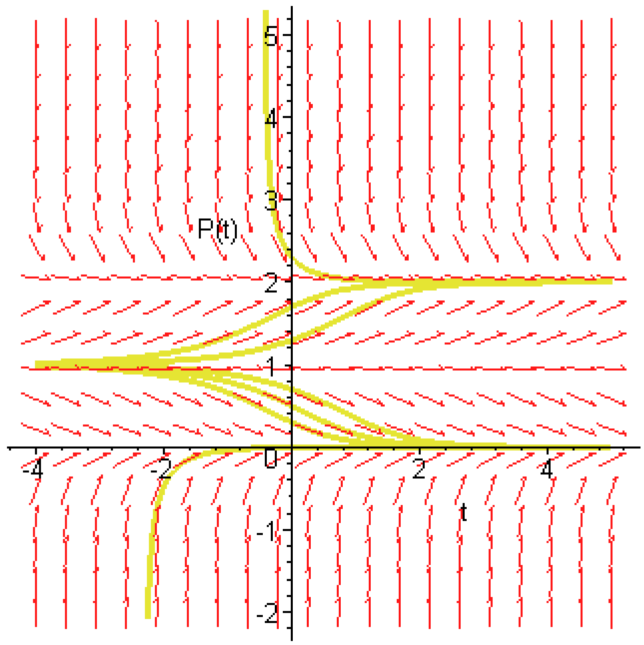
The following is another interesting autonomous ODE that is similar to the logistic equation.

$$\frac{dP}{dt} = -P(1-P)(2-P)$$

Let's draw a slope field and some solution curves.

```
> ode := diff(P(t),t)=-P(t)*(1 - P(t))*(2 - P(t));
  DEplot( ode, P(t), t=-4..5, P(t)=-2..5, [[P(0)=-0.005], [P(0)=0.3], [P(0)=0.5], [P(0)=0.7], [P(0)=
  [P(0)=1.7], [P(0)=2.3]], stepsize=0.01 );
```

$$ode := \frac{d}{dt} P(t) = -P(t)\,(1 - P(t))\,(2 - P(t))$$



Some solutions approach 2 and others approach 0. What is the dividing line between these sets of solutions? What has caused this to happen?

Look up the concepts of a threshold population size and a carrying capacity. See how those concepts apply here.

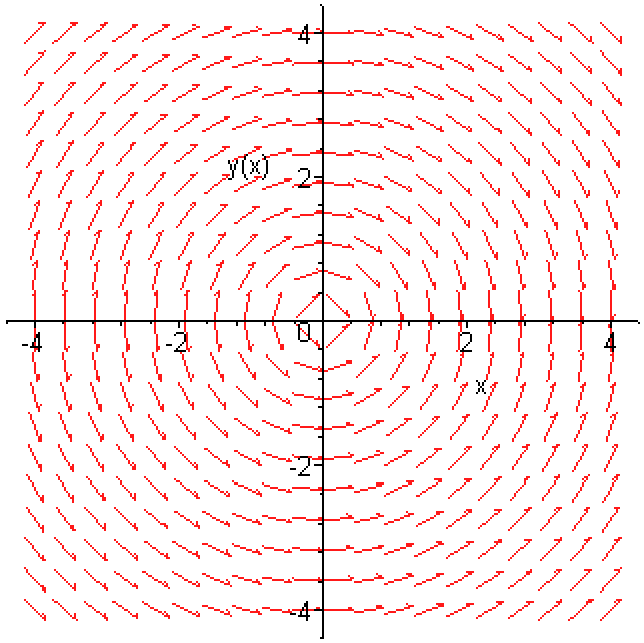
You Try It: General

Consider some nonautonomous differential equations of the form $\frac{dy}{dx} = f(x,y)$. In this section we will start with the **DEplot** command to plot the slope fields, and we will ask you to sketch some representative solution curves. Here are a few examples. See if you can predict what the solution curves will look like by looking at the direction fields. Describe the long-term behavior of the solutions for all possible initial values of $y(0)$.

Solve $\frac{dy}{dx} = -\frac{x}{y}$

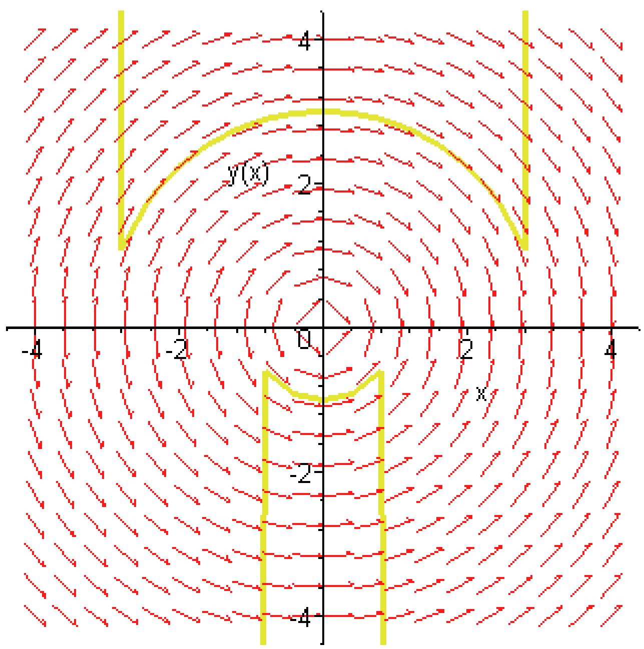
```
> ode := diff(y(x),x)=-x/y(x);
  DEplot( ode, y(x), x=-4..4, y(x)=-4..4 );
```

$$ode := \frac{d}{dx}\,y(x) = -\frac{x}{y(x)}$$



Change the function to whatever you wish. You can expect some strange looking graphs. Why might you predict this?

```
> DEplot( ode, y(x), x=-4..4, y(x)=-4..4, [[y(0)=-1], [y(0)=3]] );
```

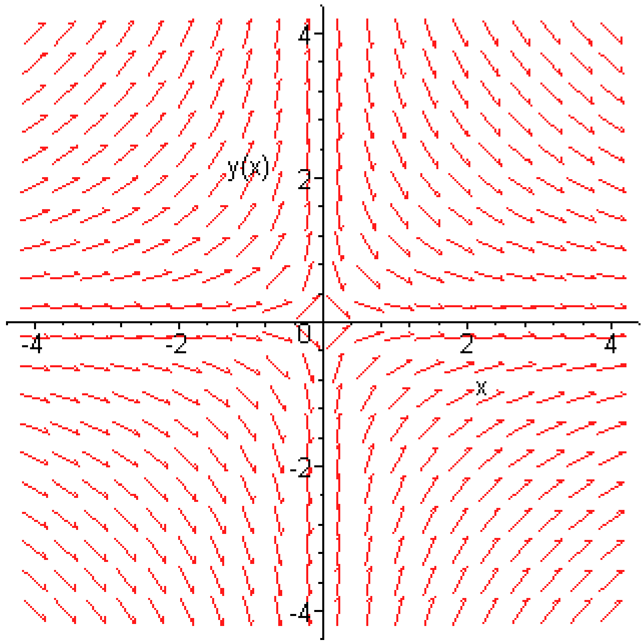


Solve $\frac{dy}{dx} = -\frac{y}{x}$

```
> ode := diff(y(x),x)=-y(x)/x;
```

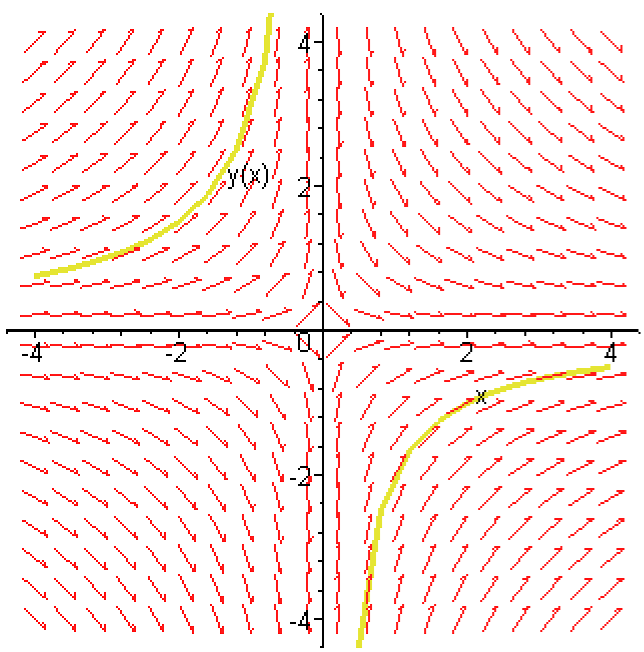
```
DEplot( ode, y(x), x=-4..4, y(x)=-4..4 );
```

$$ode := \frac{d}{dx}\,y(x) = -\frac{y(x)}{x}$$



Change the terms in the initial conditions to whatever you wish. Why would you avoid intial conditions at $x = 0$?

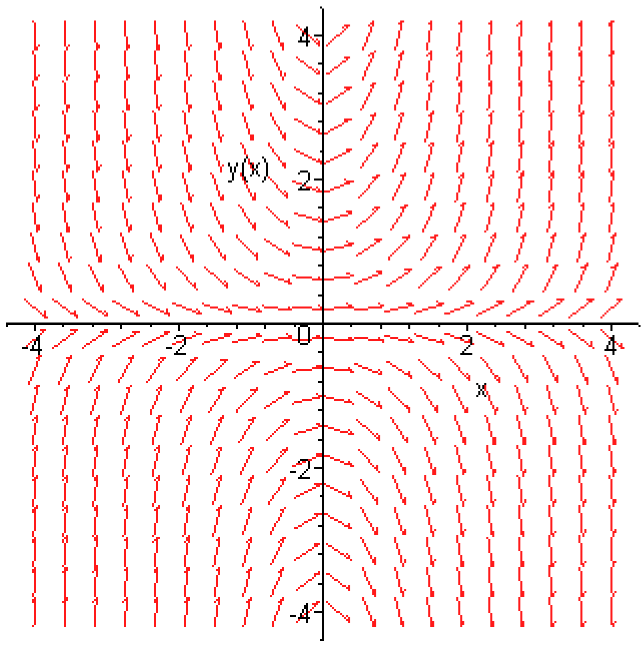
```
> DEplot( ode, y(x), x=-4..4, y(x)=-4..4, [[y(1)=-2], [y(-1)=3]] );
```



Solve $\frac{dy}{dx} = xy$

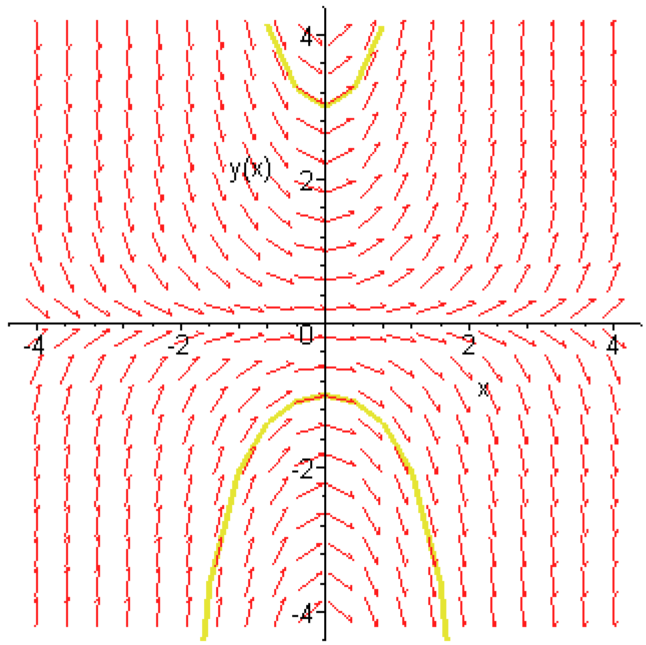
```
> ode := diff(y(x),x)=x*y(x);
  DEplot( ode, y(x), x=-4..4, y(x)=-4..4 );
```

$$ode := \frac{d}{dx}y(x) = x\,y(x)$$



Change the initial conditions to whatever you want. Add more initial conditions to see what happens.

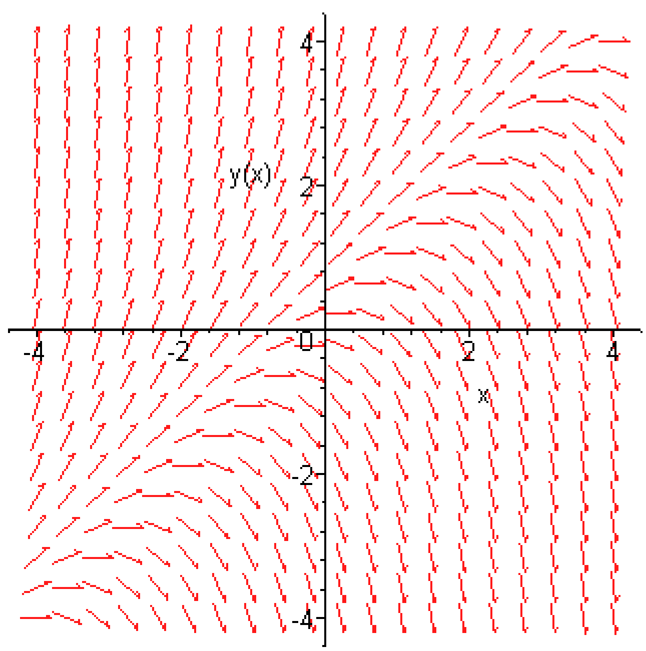
```
> DEplot( ode, y(x), x=-4..4, y(x)=-4..4, [[y(0)=-1], [y(0)=3]] );
```



Solve $\frac{dy}{dx} = y - x$

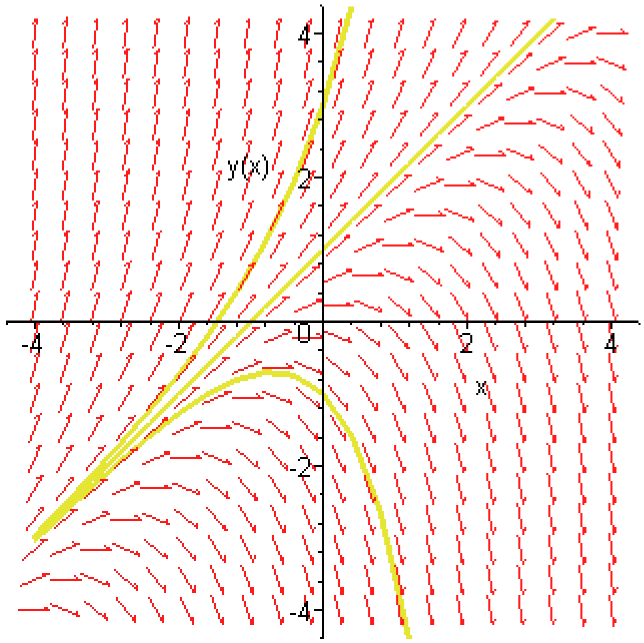
```
> ode := diff(y(x),x)=y(x) - x;
  DEplot( ode, y(x), x=-4..4, y(x)=-4..4 );
```

$$ode := \frac{d}{dx}\,y(x) = y(x) - x$$



Change the terms in the initial conditions to whatever you wish.

```
> DEplot( ode, y(x), x=-4..4, y(x)=-4..4, [[y(0)=-1], [y(0)=3], [y(0)=1]] );
```



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Try More of Your Own

It's fun! Make up some of your own differential equations of the form $\frac{dy}{dx} = f(x,y)$, and see what kind of patterns and solutions you can generate?