Moving in Three Dimensions

Note: You may notice differences between this Maple worksheet and the equivalent Mathematica notebook. These differences were introduced to preserve the content of these modules and were necessary because of major functional differences between Maple and Mathematica.

Introduction

OBJECTIVE: Learn to use *Maple* to perform calculations to analyze motion in the equations that are given parameterically.

Moving in three dimensions is something with which we are familiar, but visualizing equations of motion and computations involved in analyzing that motion can be cumbersome. This module gives a broad overview on the analysis of motion in the equations that are given parametrically.

Technology Guidelines

NOTE: If you have just finished a worksheet, **restart** *Maple* before executing a new worksheet.
TO OPEN SECTIONS,
Click on the **PLUS** sign at the left hand side of the screen *or* select **Expand All Sections** from the **View** drop down menu.

TO STOP AN EXECUTION
Click on **STOP** button from the toolbar.

ORDER OF EXECUTION
Execute commands in the order given. Do not skip any *Maple* Input lines within a given worksheet

Alternatively, you can execute the entire worksheet by selecting the **Execute Worksheet** command from the **Edit** drop down menu.

SAVING WORKSHEETS.

You can save anytime to any directory you choose, and it is wise to save often.
EXPERIENCING MAJOR PROBLEMS
Save if appropriate, and then shut down *Maple* and start it up again.

Part I: Parametric Equations of a Curve in Three Dimensions

First, we load the **plots** and **plottools** packages and the define x, t, y, and z coordinates for motion parametrically.

```
> restart:
```

```
with(plots):
with(plottools):
```

Warning, the name changecoords has been redefined

Warning, the assigned name arrow now has a global binding

```
> x:=t-> cos(t):x(t);
  y:=t-> sin(t):y(t);
  z:=t-> 4 - t^2/25;
```

$$\cos(t)$$

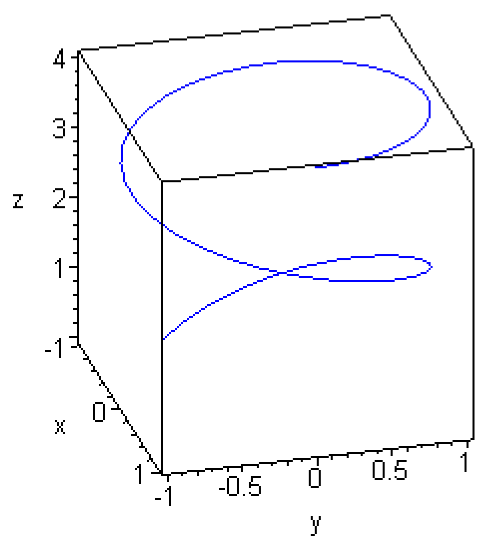
$$\sin(t)$$

$$z := t \to 4 - \frac{1}{25}t^2$$

Next, we plot the resulting curve in blue. Can you tell in which direction you are moving on the curve as t increases?

Once the graph is plotted, click and drag the plot to view it from different angles. You can do this with any *Maple* 3D plot.

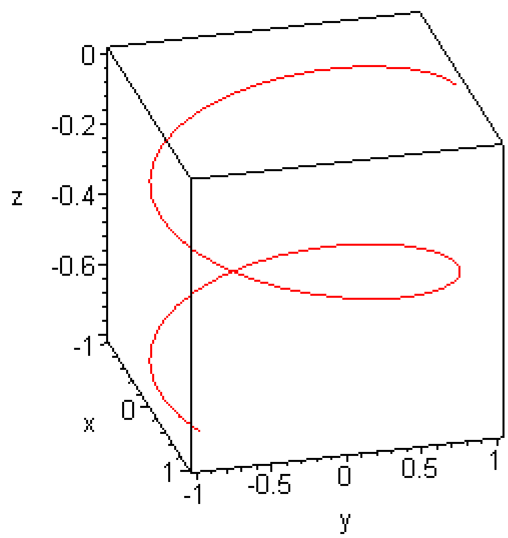
```
> p:=spacecurve([x(t),y(t),z(t),t=0..10],axes=boxed,labels=["x","y","z"],orientation=[-15,65],color=blue):
  display(p);
```



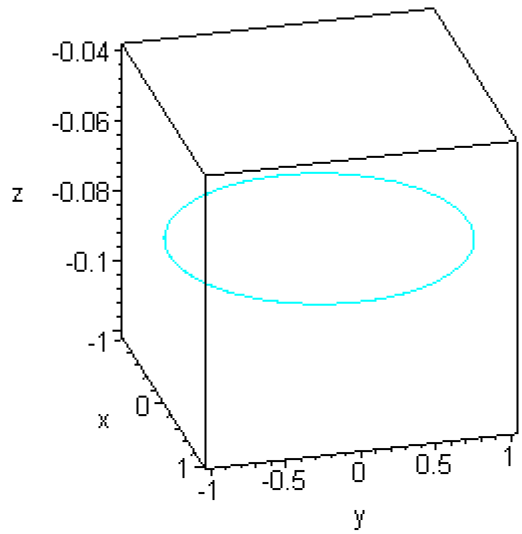
If you consider the parametric equation as a vector equation for the motion of a particle, the derivative of that vector is the velocity vector that we form by differentiating each component. The commands below plot both the first and second derivative of the vector position function.

```
> p1:=spacecurve([diff(x(t),t),diff(y(t),t),diff(z(t),t),t=0..10],axes=boxed,labels=
["x","y","z"],orientation=[-15,65],color=COLOR(RGB,1,0,0)):
p2:=spacecurve([diff(x(t),t$2),diff(y(t),t$2),diff(z(t),t$2),t=0..10],axes=boxed,labels=
["x","y","z"],orientation=[-15,65],color=COLOR(RGB,0,1,1)):
print(cat(`The velocity vector is `,[diff(x(t),t),diff(y(t),t),diff(z(t),t)]));
display(p1);
print(cat(`The acceleration vector is `,[diff(x(t),t$2),diff(y(t),t$2),diff(z(t),t$2)]));
display(p2);
```

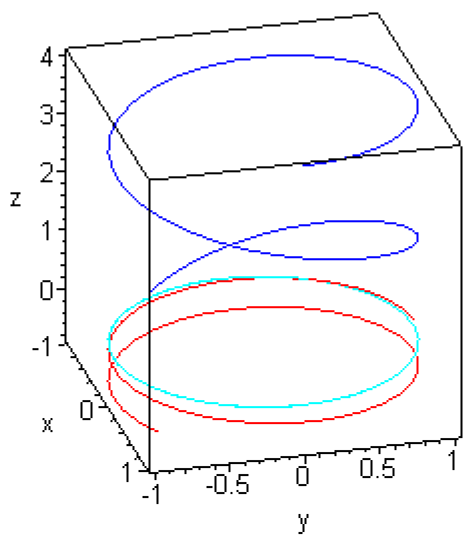
$$\textit{The velocity vector is } \| \left[-\sin(t), \cos(t), -\frac{2\,t}{25} \right]$$



$$\textit{The acceleration vector is } \| \left[-\cos(t), -\sin(t), \frac{-2}{25} \right]$$



> **display(p,p1,p2);**

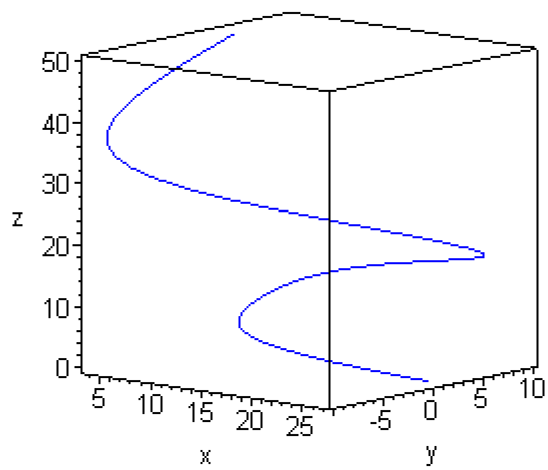


You Try It: Part I

Define your own *x*, *y*, and *z* coordinates for motion by changing the entries for **x, y, z.** You may or may not want to change the time interval (**timeinterval**) over which you plot your functions.

```
> x:=t-> 10* exp(cos(t)):
  y:=t-> 10* sin(t^2/15):
  z:=t-> t^2/2:
  timeinterval:= 5:
```

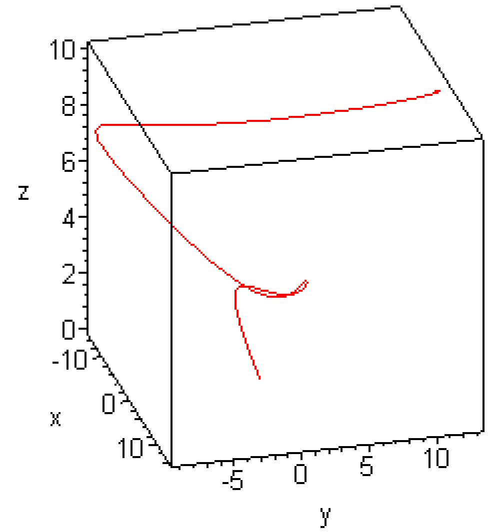
p:=spacecurve([x(t),y(t),z(t)],t=0..10,axes=boxed,color=COLOR(RGB,0,0,1),orientation=[-50,80], labels=["x","y","z"]):display(p);



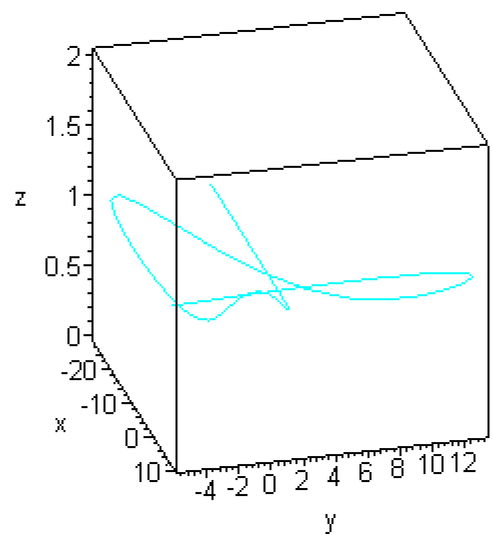
Now check out the velocity and acceleration

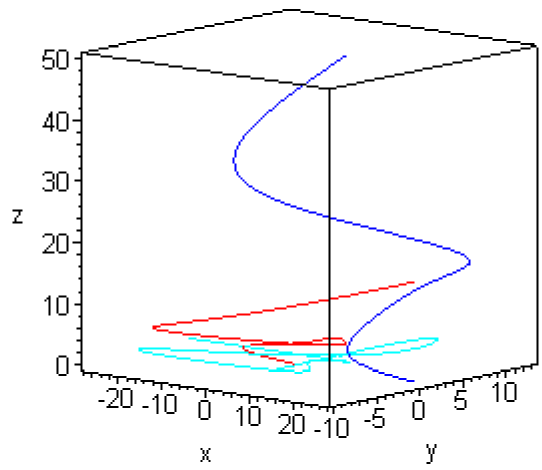
```
> p1:=spacecurve([diff(x(t),t),diff(y(t),t),diff(z(t),t),t=0..10],axes=boxed,labels=
["x","y","z"],orientation=[-15,65],color=COLOR(RGB,1,0,0)):
p2:=spacecurve([diff(x(t),t$2),diff(y(t),t$2),diff(z(t),t$2),t=0..10],axes=boxed,labels=
["x","y","z"],orientation=[-15,65],color=COLOR(RGB,0,1,1)):

> print(cat(`The velocity vector is `,[diff(x(t),t),diff(y(t),t),diff(z(t),t)]));
display(p1);
print(cat(`The acceleration vector is `,[diff(x(t),t$2),diff(y(t),t$2),diff(z(t),t$2)]));
display(p2);
display(p,p1,p2);
```

$$\textit{The velocity vector is } \|\left[-10\sin(t)\,\mathbf{e}^{\cos(t)}, \frac{4}{3}\cos\left(\frac{t^2}{15}\right)t, t\right]$$



The acceleration vector is $\left\|\left[-10\cos(t)\,\mathrm{e}^{\cos(t)}+10\sin(t)^2\,\mathrm{e}^{\cos(t)},-\frac{8}{45}\sin\left(\frac{t^2}{15}\right)t^2+\frac{4}{3}\cos\left(\frac{t^2}{15}\right.\right.\right.$





The plot of all three together may or may not be instructive.

Compare your results to those of your classmates.

Part II: Equations of Motion *Velocity* $\rightarrow$ *Position*

Velocity $\rightarrow$ *Acceleration*

If you are given the velocity, you can differentiate to find the acceleration and integrate to find the displacement. Since we do three integrations, we need three arbitrary constants to get the general solution.

```
> unassign('t,a,b,c');
```

```
> velocity:= t-> [3*sqrt(t+1)/2,exp(-t),1/(t+1)]:
  acceleration:= t-> diff(velocity(t),t):
  ([seq(int(velocity(t)[i],t),i=1..nops(velocity(t)))] ):
  % + [a,b,c]:
  rgeneral:= %:
```

```
> print(cat(`The velocity vector is defined to be `,velocity(t)));
  print(cat(`The acceleration vector is `,acceleration(t)));
  print(cat(`The position vector is `,rgeneral));
```

$$\textit{The velocity vector is defined to be } \left\| \left[\frac{3\sqrt{t+1}}{2}, e^{(-t)}, \frac{1}{t+1} \right] \right.$$

$$\textit{The acceleration vector is } \left\| \left[\frac{3}{4\sqrt{t+1}}, -e^{(-t)}, -\frac{1}{(t+1)^2} \right] \right.$$

$$\textit{The position vector is } \left\| \left[a + (t+1)^{\left(\frac{3}{2}\right)}, b - e^{(-t)}, c + \ln(t+1) \right] \right.$$

```
> s:=eval(rgeneral, t=0);
```

$$s := [a+1, b-1, c]$$

```
> initial:=[1,1,0]:
  eqns:={seq(s[i]=initial[i],i=1..3)};
  assign(solve(eqns,{a,b,c}));
```

$$eqns := \{a+1=1, b-1=1, c=0\}$$

```
> r:=rgeneral;
```

$$r := \left[(t+1)^{\left(\frac{3}{2}\right)}, 2 - e^{(-t)}, \ln(t+1) \right]$$

You Try It: Part II

Change the entries in your velocity vector and your initial position (**velocity** and **initial**), and re-execute the section.

```
> unassign('a,b,c,t');
  velocity:=[sqrt(t+1)*3/2,exp(-t),1/(t+1)]:
  initial:=[1,1,0]:
  acceleration:=diff(velocity,t):
  [seq(int(velocity[i],t),i=1..nops(velocity))]:
  %+[a,b,c]:
  rgeneral:=%:
  s:=subs(t=0,rgeneral):
  eqns:={seq(subs(s[i]=initial[i]),i=1..3)}:
  assign(solve(eqns,{a,b,c})):
  r:=rgeneral;
```

$$r := \left[(t+1)^{\left(\frac{3}{2}\right)}, 2 - e^{(-t)}, \ln(t+1) \right]$$

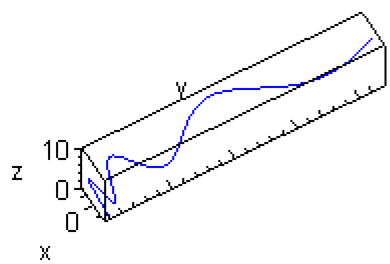
Part III: Computing the Distance Traveled on a Curved Path

First we define the x, y, and z coordinates for motion parametrically.

```
> unassign('r,t,v');
```

```
> r:=[3*cos(2*t),t^(10/3)/50,t]:
  v:=diff(r,t):
  speed:=(sqrt(linalg[dotprod](v,v))):
  p:=spacecurve(r(t),t=0..10,axes=boxed,labels=["x","y","z"],orientation=[-15,65],color=blue,orientation=[-30,33],scaling=constrained):
  distance:=evalf(int(speed,t=0..10)):
  print(`The position given in feet is `,r);
  display(p);
  print(`The velocity vector given in feet per minute is `,v);
  print(`The speed `,ds/dt,` given in feet per minute is `,speed);
  print(`The distance traveled in feet in 10 minutes is `,distance);
```

$$\textit{The position given in feet is}, \left[3\cos(2\,t), \frac{t^{\left(\frac{10}{3}\right)}}{50}, t\right]$$



$$\textit{The velocity vector given in feet per minute is}, \left[-6\sin(2\,t), \frac{t^{\left(\frac{7}{3}\right)}}{15}, 1\right]$$

$$\textit{The speed}\ ,\ \frac{ds}{dt},\ \textit{given in feet per minute is}\ ,\ \frac{1}{15}\sqrt{225+8100\sin(2\,t)\sin(2\,\bar{t})+t^{\left(\frac{7}{3}\right)}\,\overline{t^{\left(\frac{7}{3}\right)}}}$$

$$\textit{The distance traveled in feet in 10 minutes is}\ ,\ 66.02703743$$

In the above example, if you had tried to integrate symbolically, instead of numerically, what would have happened? (The **f** after **eval** signifies numerical integration.) Many integrals arising in the computation of arc length are too complicated to evaluate symbolically.

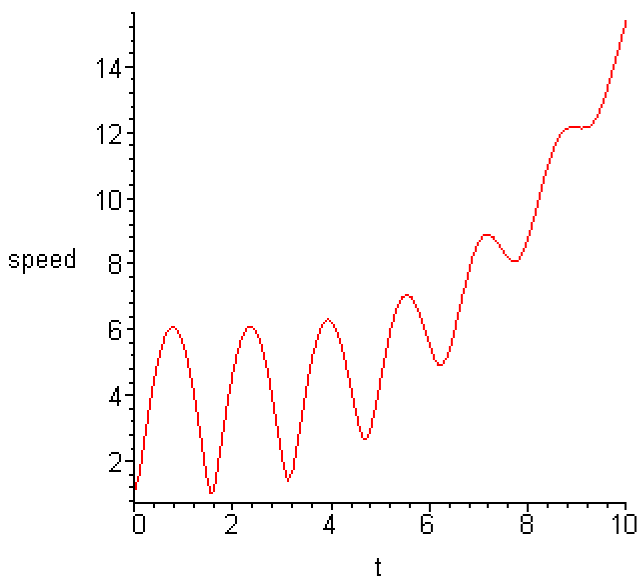
```
> int(speed,t=0..10);
```

$$\int_0^{10}\frac{1}{15}\sqrt{225+8100\sin(2\,t)\sin(2\,\bar{t})+t^{\left(\frac{7}{3}\right)}\,\overline{t^{\left(\frac{7}{3}\right)}}}\,dt$$

```
> print(`Speed at 5 minutes is `,evalf(subs(t=5,speed),6),` feet per minute`);
```

$$\textit{Speed at 5 minutes is}\ ,\ 4.44712,\ \textit{feet per minute}$$

```
> plot(speed,t=0..10,labels=[t,"speed"]);
```



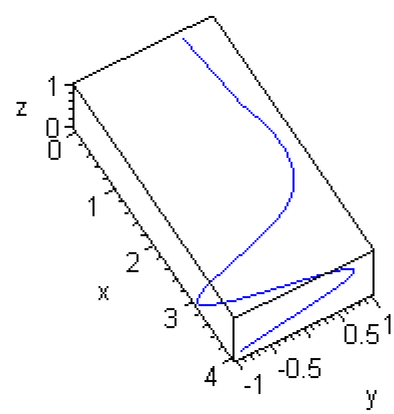
If *t* represents time in minutes, what does this say about the speed of the hiker? Is it possible?

You Try It: Part III

Choose your own position function (**r**) and the interval over which you wish to integrate.

```
> unassign('r,t,v,speed,distance');
r:=[2*t^(.3),cos(t-1),exp(-t*t)]:
v:=diff(r,t):
speed:=simplify((sqrt(linalg[dotprod](v,v)))):
p:=spacecurve(r(t),t=0..10,axes=boxed,labels=["x","y","z"],orientation=[-15,65],color=blue,orientation=[-30,33],scaling=constrained):
distance:=int(speed,t=0..10):
print(`The position given in feet is `,r);
display(p);
print(`The velocity vector given in feet per minute is `,v);
print(`The speed `,ds/dt,` given in feet per minute is `,speed);
print(`The distance traveled in feet in 10 minutes is `,distance);
```

The position given in feet is , $\left[2\,t^{0.3},\ \cos(t-1),\ e^{(-t^2)}\right]$



The velocity vector given in feet per minute is , $\left[\frac{0.6}{t^{0.7}},\ -\sin(t-1),\ -2\,t\,e^{(-t^2)}\right]$

The speed , $\frac{ds}{dt}$, *given in feet per minute is* ,

$$\frac{0.2000000000\sqrt{9. + 25.\ \sin(t - 1.)\ \sin\left(\frac{-1.\ t + |t|^2}{t}\right) |t|^{\left(\frac{7}{5}\right)} + 100.\ \mathrm{e}^{(-2.\ \Re(t^2))} |t|^{\left(\frac{17}{5}\right)}}}{|t|^{\left(\frac{7}{10}\right)}}$$

The distance traveled in feet in 10 minutes is ,

$$\int_0^{10} \frac{0.2000000000\sqrt{9. + 25.\ \sin(t - 1.)\ \sin\left(\frac{-1.\ t + |t|^2}{t}\right) |t|^{\left(\frac{7}{5}\right)} + 100.\ \mathrm{e}^{(-2.\ \Re(t^2))} |t|^{\left(\frac{17}{5}\right)}}}{|t|^{\left(\frac{7}{10}\right)}}$$

You will notice that Maple was unable to evaluate the integral for the distance function symbolically. In Version 6, it also fails to evaluate the integral numerically. This could be a glitch that will be fixed soon.

You can check it out by executing the following command, but you will probably have to hit the STOP button.

```
> distance:=evalf(int(speed,t=0..10)):
```

To see the evaluation of the integral above, either use Version 5.1 of Maple or go to the *Mathematica* version on your CD.

Part IV: Computing Curvature and Torsion for a Space Curve

Maple simplifies the process of finding curvature and torsion. The computations below use formulas directly from your text, and we graphically explore the interpretations of the curvature and torsion functions.

```
> with(linalg):
```

Warning, the protected names norm and trace have been redefined and unprotected

```
> unassign('r,v,t,speed');
  mag:=proc(v) RETURN(sqrt(linalg[dotprod](v,v))): end :
  r:=[10*sin(t),10*exp(-t),t^(10/3)/100]:
  v:=diff(r,t):
```

```
a:=diff(v,t):
speed:=mag(v):
utan:=[v[1]/speed,v[2]/speed,v[3]/speed]:
curvature:=simplify(mag(linalg[crossprod](v,a))/speed^3):          torsion:=simplify(det(ma
(3,3,[v,a,diff(a,t)]))/simplify(multiply(crossprod(v,a),crossprod(v,a)))):
print(`The position vector is `, r):
print(`The velocity vector is `, v):
print(`The acceleration vector is `, a):
print(`The speed is `, speed):
print(`The unit tangent vector is `, utan):
print(`The curvature is `, curvature):
print(`The torsion is `, torsion):
```

$$\textit{The position vector is }, \left[10\sin(t),\ 10\,e^{(-t)},\ \frac{t^{\left(\frac{10}{3}\right)}}{100}\right]$$

$$\textit{The velocity vector is }, \left[10\cos(t),\ -10\,e^{(-t)},\ \frac{t^{\left(\frac{7}{3}\right)}}{30}\right]$$

$$\textit{The acceleration vector is }, \left[-10\sin(t),\ 10\,e^{(-t)},\ \frac{7\,t^{\left(\frac{4}{3}\right)}}{90}\right]$$

$$\textit{The speed is }, \frac{1}{30}\sqrt{90000\cos(t)\cos(\bar{t}) + 90000\,e^{(-t)}\,e^{(-\bar{t})} + t^{\left(\frac{7}{3}\right)}\,\overline{\left(t^{\left(\frac{7}{3}\right)}\right)}}$$

$$\textit{The unit tangent vector is }, \left[\frac{300\cos(t)}{\sqrt{90000\cos(t)\cos(\bar{t}) + 90000\,e^{(-t)}\,e^{(-\bar{t})} + t^{\left(\frac{7}{3}\right)}\,\overline{\left(t^{\left(\frac{7}{3}\right)}\right)}}},\right.$$

$$-\frac{300\,e^{(-t)}}{\sqrt{90000\cos(t)\cos(\bar{t}) + 90000\,e^{(-t)}\,e^{(-\bar{t})} + t^{\left(\frac{7}{3}\right)}\,\overline{\left(t^{\left(\frac{7}{3}\right)}\right)}}},$$

$$\frac{t^{\left(\frac{7}{3}\right)}}{\sqrt{90000\cos(t)\cos(\bar{t}) + 90000\,e^{(-t)}\,e^{(-\bar{t})} + t^{\left(\frac{7}{3}\right)}\,\overline{\left(t^{\left(\frac{7}{3}\right)}\right)}}}\Bigg]$$

The curvature is , $3000\,I\sqrt{810000\,e^{(-2\,\Re(t))}\left|-\cos(t)+\sin(t)\right|^2 + \left|t^{\left(\frac{8}{3}\right)}\,(3\,t\sin(t)+7\cos(t))^2\right| + e^{(-2\,\Re(t))}\left|t^{\left(\frac{8}{3}\right)}\right.}$

$$\Bigg/\left(90000\cos(t)\cos\left(\frac{|t|^2}{t}\right) + 90000\,e^{\left(-\frac{t^2+|t|^2}{t}\right)} + |t|^{\left(\frac{14}{3}\right)}\right)^{\left(\frac{3}{2}\right)}$$

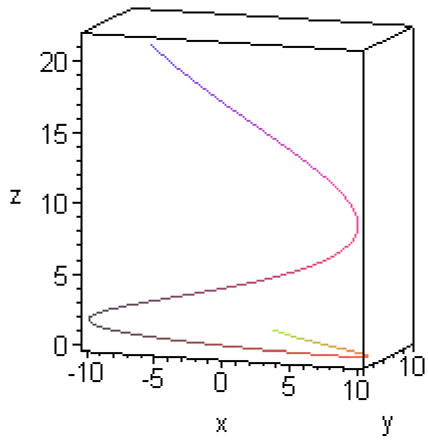
The torsion is , $-30\,e^{(-t)}\,t^{\left(\frac{1}{3}\right)}\,(28\cos(t)+42\cos(t)\,t-28\sin(t)+9\sin(t)\,t^2+9\cos(t)\,t^2)\,/$

$$\Big(-49\,e^{(-2\,t)}\,t^{\left(\frac{8}{3}\right)} - 42\,e^{(-2\,t)}\,t^{\left(\frac{11}{3}\right)} - 9\,t^{\left(\frac{14}{3}\right)}\,e^{(-2\,t)} - 9\,t^{\left(\frac{14}{3}\right)} + 9\,t^{\left(\frac{14}{3}\right)}\cos(t)^2 - 42\,t^{\left(\frac{11}{3}\right)}\,s$$

$$-\,49\,t^{\left(\frac{8}{3}\right)}\cos(t)^2 + 1620000\cos(t)\,e^{(-2\,t)}\sin(t) - 810000\,e^{(-2\,t)}\Big)$$

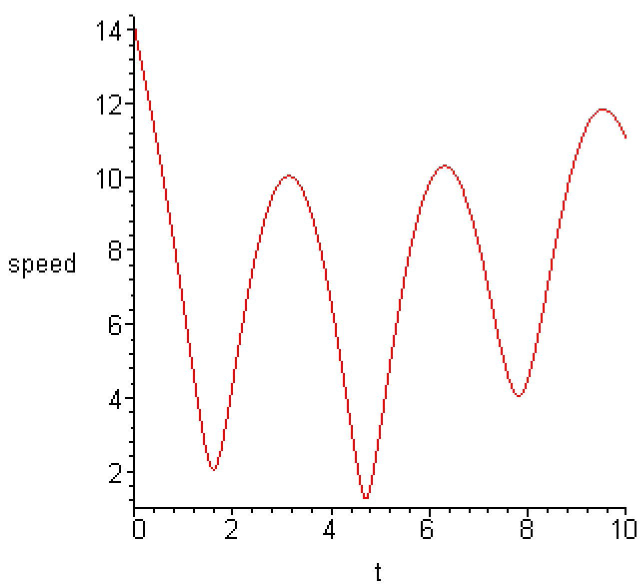
Let's visualize some of these quantities. We will begin by drawing the path of motion:

```
> p:=spacecurve(r,t=0..10,axes=boxed,orientation=[-70,80],labels=[x,y,z], scaling=constraine
  display(p);
```




```
> print(`The speed is `,speed);
  ps:=plot(speed,t=0..10,labels=["t","speed"]):
  display(ps);
```

$$\text{The speed is },\ \frac{1}{30}\sqrt{90000\cos(t)\cos(\bar{t})+90000\,\mathbf{e}^{(-t)}\,\mathbf{e}^{(-\bar{t})}+t^{\left(\frac{7}{3}\right)}\overline{\left(t^{\left(\frac{7}{3}\right)}\right)}}$$




```
> print(`The curvature is `,curvature);
  pc:=plot(curvature,t=0..10,color=green, labels=["t","curvature"]):
  display(pc);
```

The curvature is , 3000 I

$$\sqrt{810000\, e^{(-2\,\Re(t))} \left|-\cos(t)+\sin(t)\right|^2 + \left|t^{\left(\frac{8}{3}\right)} (3\,t\,\sin(t)+7\,\cos(t))^2\right| + e^{(-2\,\Re(t))} \left|t^{\left(\frac{8}{3}\right)}\right.}$$
$$\Big/ \left(90000\cos(t)\cos\left(\frac{|t|^2}{t}\right) + 90000\, e^{\left(-\frac{t^2+|t|^2}{t}\right)} + |t|^{\left(\frac{14}{3}\right)}\right)^{\left(\frac{3}{2}\right)}$$

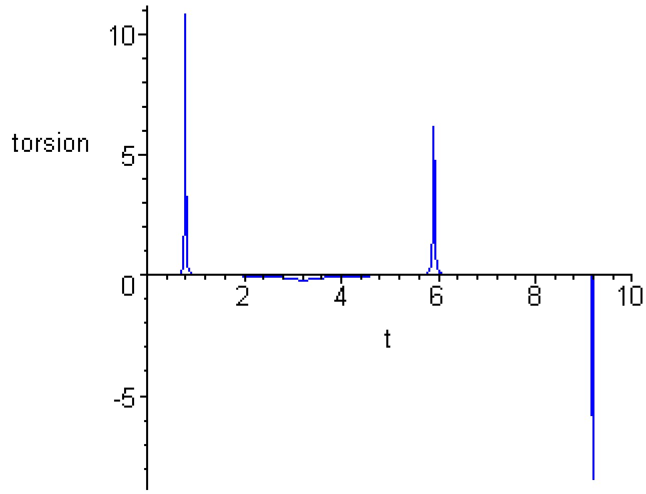
Warning, unable to evaluate the function to numeric values in the region; see the plotting command's help page to ensure the calling sequence is correct

Plotting error, empty plot

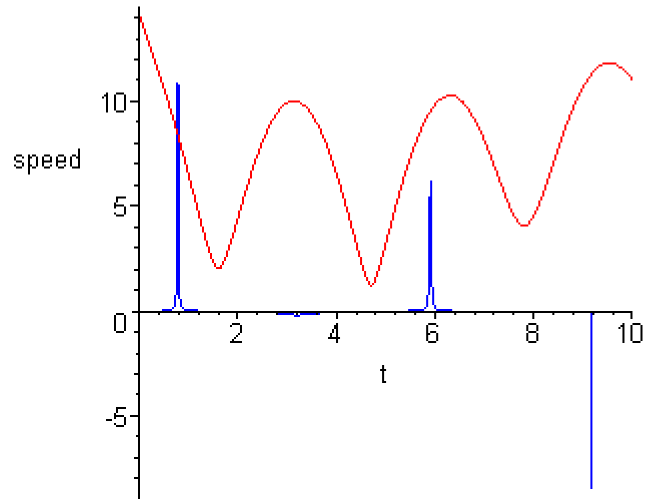
```
> print(`The torsion is `,torsion);
  pt:=plot(torsion,t=0..10,color=blue, labels=["t","torsion"]):
  display(pt);
```

The torsion is , $-30\, e^{(-t)}\, t^{\left(\frac{1}{3}\right)} (28\cos(t)+42\cos(t)\,t-28\sin(t)+9\sin(t)\,t^2+9\cos(t)\,t^2)$ /

$$-49\, e^{(-2\,t)}\, t^{\left(\frac{8}{3}\right)} - 42\, e^{(-2\,t)}\, t^{\left(\frac{11}{3}\right)} - 9\, t^{\left(\frac{14}{3}\right)} e^{(-2\,t)} - 9\, t^{\left(\frac{14}{3}\right)} + 9\, t^{\left(\frac{14}{3}\right)} \cos(t)^2 - 42\, t^{\left(\frac{11}{3}\right)} \text{s}$$
$$- 49\, t^{\left(\frac{8}{3}\right)} \cos(t)^2 + 1620000\cos(t)\, e^{(-2\,t)} \sin(t) - 810000\, e^{(-2\,t)}\Big)$$




```
> display(ps,pc,pt);
  print(`Speed in Red, curvature in Green, torsion in Blue`);
```



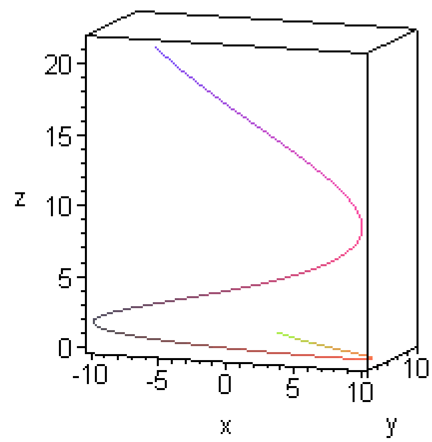
Speed in Red, curvature in Green, torsion in Blue

Look at your space curve and see if you can identify the points on your space curve at which the curvature or torsion spike. What has caused these events? Are they related at all to the speed?

Recall from the definition of curvature that relates it to the cross product of the velocity and acceleration that the curvature is smallest when the velocity and acceleration are in the same direction. The curvature is largest when the velocity and acceleration are perpendicular to one another.

Looking at the torsion as the dot product of the derivative of the unit binormal and the normal vectors. Torsion will peak when those two vectors are in the same direction and be close to 0 when those two vectors are perpendicular.

```
> display(p);
```



You can see the TNB frame for a curve in action in Part VI of this module.

You Try It: Part IV

Redefine your position function (**r**) and compute. It should be noted that the computations here are not trivial, so more complicated functions could lead to long waits.

```
> unassign('r,v,t,speed');
  r:=[2*t^.3,cos(t-1),exp(-t*t)]:
  mag:=proc(v) RETURN(sqrt(dotprod(v,v))): end :
  v:=diff(r,t):
  a:=diff(v,t):
  speed:=mag(v):
  utan:=[v[1]/speed,v[2]/speed,v[3]/speed]:
  curvature:=simplify(mag(linalg[crossprod](v,a))/speed^3):torsion:=simplify(det(matrix(3,3
  [v,a,diff(a,t)]))/simplify(multiply(crossprod(v,a),crossprod(v,a)))):
  print(`The position vector is `, r):
  print(`The velocity vector is `, v):
  print(`The acceleration vector is `, a):
  print(`The speed is `, speed):
```

print(`The unit tangent vector is `, utan):
print(`The curvature is `, curvature):
print(`The torsion is `, torsion):

$$\textit{The position vector is },\left[2\,t^{0.3},\ \cos(t-1),\ \mathbf{e}^{(-t^2)}\right]$$

$$\textit{The velocity vector is },\left[\frac{0.6}{t^{0.7}},\ -\sin(t-1),\ -2\,t\,\mathbf{e}^{(-t^2)}\right]$$

$$\textit{The acceleration vector is },\left[-\frac{0.42}{t^{1.7}},\ -\cos(t-1),\ -2\,\mathbf{e}^{(-t^2)}+4\,t^2\,\mathbf{e}^{(-t^2)}\right]$$

$$\textit{The speed is },\sqrt{\frac{0.36\,\overline{\left(\frac{1}{t^{0.7}}\right)}}{t^{0.7}}+\sin(t-1)\,\sin(-1+\bar{t})+4\,t\,\mathbf{e}^{(-t^2)}\,\overline{\left(t\,\mathbf{e}^{(-t^2)}\right)}}$$

$$\textit{The unit tangent vector is },\left[\frac{0.6}{t^{0.7}\sqrt{\frac{0.36\,\overline{\left(\frac{1}{t^{0.7}}\right)}}{t^{0.7}}+\sin(t-1)\,\sin(-1+\bar{t})+4\,t\,\mathbf{e}^{(-t^2)}\,\overline{\left(t\,\mathbf{e}^{(-t}\right.}}}\right.$$

$$-\frac{\sin(t-1)}{\sqrt{\frac{0.36\,\overline{\left(\frac{1}{t^{0.7}}\right)}}{t^{0.7}}+\sin(t-1)\,\sin(-1+\bar{t})+4\,t\,\mathbf{e}^{(-t^2)}\,\overline{\left(t\,\mathbf{e}^{(-t^2)}\right)}}},$$

$$\left.-\frac{2\,t\,\mathbf{e}^{(-t^2)}}{\sqrt{\frac{0.36\,\overline{\left(\frac{1}{t^{0.7}}\right)}}{t^{0.7}}+\sin(t-1)\,\sin(-1+\bar{t})+4\,t\,\mathbf{e}^{(-t^2)}\,\overline{\left(t\,\mathbf{e}^{(-t^2)}\right)}}}\right]$$

The curvature is , $2.500000000 \left(36.\, e^{(-2.\, \Re(t^2))} \left| \frac{(-17. + 20.\, t^2)^2}{t^{\left(\frac{7}{5}\right)}} \right| + 9. \left| \frac{(10.\, t \cos(t - 1.) + 7.\, \sin(t - 1.))^2}{t^{\left(\frac{17}{5}\right)}} \right| + 10000.\, e^{(-2.\, \Re(t^2))} \left| -1.\, \sin(t - 1.) + 2.\, \sin(t - 1.)\, t^2 + t \cos(t - 1.) \right|^2 \right)^{\left(\frac{1}{2}\right)} |t|^{\left(\frac{21}{10}\right)} \Big/ \left(9. + 25.\, \sin(t - 1.) \sin\left(\frac{-1.\, t + |t|^2}{t} \right) |t|^{\left(\frac{7}{5}\right)} + 100.\, e^{(-2.\, \Re(t^2))} |t|^{\left(\frac{17}{5}\right)} \right)^{\left(\frac{3}{2}\right)}$

The torsion is , $-30.\, e^{(-1.\, t^2)} (-600.\, t^3 \cos(t - 1.) + 400.\, t^5 \cos(t - 1.) - 488.\, \sin(t - 1.)\, t^2 + 80.\, t^4 \sin(t - 1.) + 119.\, \sin(t - 1.) - 119.\, t \cos(t - 1.))\, t^{\left(\frac{7}{10}\right)} \Big/ \left(-10000.\, e^{(-2.\, t^2)}\, t^{\left(\frac{17}{5}\right.} \right.$

$+ 10000.\, e^{(-2.\, t^2)}\, t^{\left(\frac{17}{5}\right)} \cos(t - 1.)^2 + 40000.\, e^{(-2.\, t^2)}\, t^{\left(\frac{27}{5}\right)} - 50000.\, e^{(-2.\, t^2)}\, t^{\left(\frac{27}{5}\right)} \cos(t$

$+ 20000.\, e^{(-2.\, t^2)}\, t^{\left(\frac{22}{5}\right)} \sin(t - 1.) \cos(t - 1.) - 40000.\, e^{(-2.\, t^2)}\, t^{\left(\frac{37}{5}\right)}$

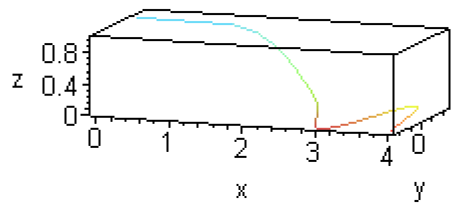
$+ 40000.\, e^{(-2.\, t^2)}\, t^{\left(\frac{37}{5}\right)} \cos(t - 1.)^2 - 40000.\, e^{(-2.\, t^2)}\, t^{\left(\frac{32}{5}\right)} \sin(t - 1.) \cos(t - 1.) - 10404.$

$+ 24480.\, e^{(-2.\, t^2)}\, t^4 - 14400.\, e^{(-2.\, t^2)}\, t^6 - 900.\, \cos(t - 1.)^2\, t^2 - 1260.\, \cos(t - 1.)\, t \sin(t - 1.$

$\left. + 441.\, \cos(t - 1.)^2 \right)$

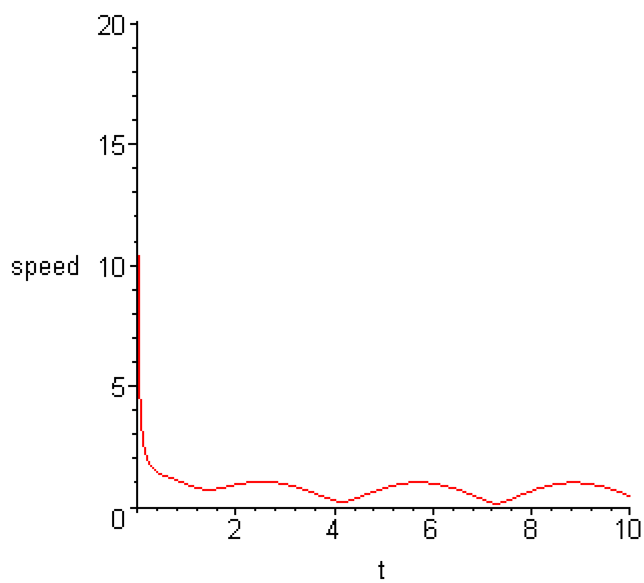
```
> p:=spacecurve(r,t=0..10,axes=boxed,orientation=[-70,80],labels=[x,y,z], scaling=constraine
```

```
display(p);
```



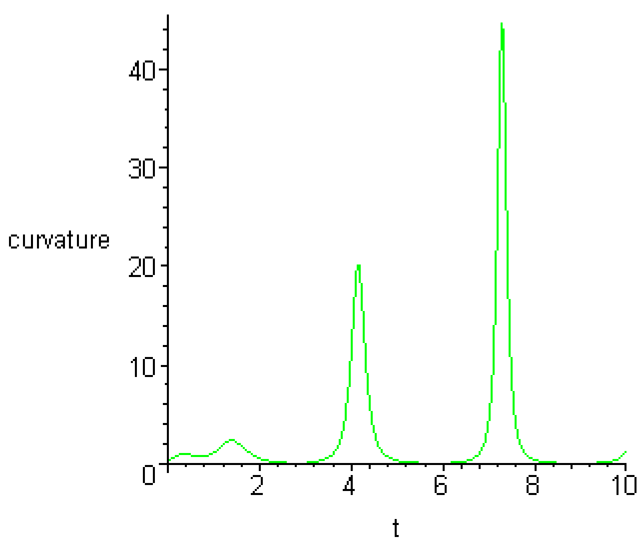

```
> print(`The speed is `,speed);
ps:=plot(speed,t=0..10,labels=["t","speed"]):
display(ps);
```

$$\textit{The speed is },\ \sqrt{\frac{0.36\left(\overline{\frac{1}{t^{0.7}}}\right)}{t^{0.7}}+\sin(t-1)\sin(-1+\bar{t})+4\,t\,\mathbf{e}^{(-t^2)}\,\overline{\left(t\,\mathbf{e}^{(-t^2)}\right)}}$$



> **print(`The curvature is `,curvature);**
> **pc:=plot(curvature,t=0..10,color=green, labels=["t","curvature"]):**
> **display(pc);**

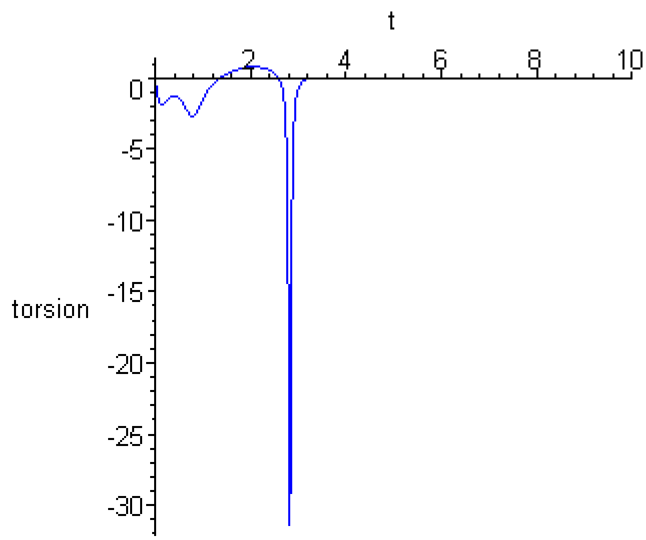
$$\textit{The curvature is}\,,\ 2.500000000 \left(36.\, \mathbf{e}^{(-2.\, \Re(t^2))} \left| \frac{(-17. + 20.\, t^2)^2}{t^{\left(\frac{7}{5}\right)}} \right| + 9. \left| \frac{(10.\, t \cos(t-1.) + 7.\, \sin(t-1.))^2}{t^{\left(\frac{17}{5}\right)}} \right| + 10000.\, \mathbf{e}^{(-2.\, \Re(t^2))} \left| -1.\, \sin(t-1.) + 2.\, \sin(t-1.)\, t^2 + t \cos(t-1.) \right|^2 \right)^{\left(\frac{1}{2}\right)} |t|^{\left(\frac{21}{10}\right)} \Big/ \left(9. + 25.\, \sin(t-1.) \sin\left(\frac{-1.\, t + |t|^2}{t} \right) |t|^{\left(\frac{7}{5}\right)} + 100.\, \mathbf{e}^{(-2.\, \Re(t^2))} |t|^{\left(\frac{17}{5}\right)} \right)^{\left(\frac{3}{2}\right)}$$



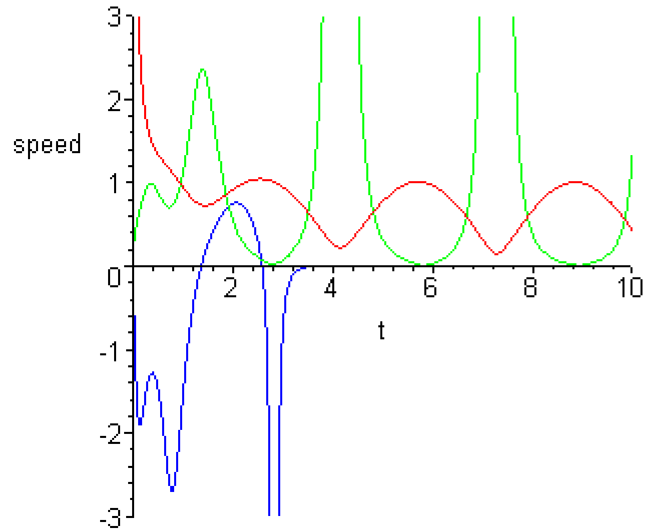
> **print(`The torsion is `,torsion);**

pt:=plot(torsion,t=0..10,color=blue,labels=["t","torsion"]):
display(pt);

The torsion is , $-30.\, e^{(-1.\, t^2)} (-600.\, t^3 \cos(t-1.) + 400.\, t^5 \cos(t-1.) - 488.\, \sin(t-1.)\, t^2$
$+ 80.\, t^4 \sin(t-1.) + 119.\, \sin(t-1.) - 119.\, t\cos(t-1.))\, t^{\left(\frac{7}{10}\right)} \Big/ \Big(-10000.\, e^{(-2.\, t^2)}\, t^{\left(\frac{17}{5}\right)}$
$+ 10000.\, e^{(-2.\, t^2)}\, t^{\left(\frac{17}{5}\right)} \cos(t-1.)^2 + 40000.\, e^{(-2.\, t^2)}\, t^{\left(\frac{27}{5}\right)} - 50000.\, e^{(-2.\, t^2)}\, t^{\left(\frac{27}{5}\right)} \cos(t$
$+ 20000.\, e^{(-2.\, t^2)}\, t^{\left(\frac{22}{5}\right)} \sin(t-1.) \cos(t-1.) - 40000.\, e^{(-2.\, t^2)}\, t^{\left(\frac{37}{5}\right)}$
$+ 40000.\, e^{(-2.\, t^2)}\, t^{\left(\frac{37}{5}\right)} \cos(t-1.)^2 - 40000.\, e^{(-2.\, t^2)}\, t^{\left(\frac{32}{5}\right)} \sin(t-1.) \cos(t-1.) - 10404.$
$+ 24480.\, e^{(-2.\, t^2)}\, t^4 - 14400.\, e^{(-2.\, t^2)}\, t^6 - 900.\, \cos(t-1.)^2\, t^2 - 1260.\, \cos(t-1.)\, t \sin(t-1.$
$+ 441.\, \cos(t-1.)^2 \Big)$



> **display(ps,pc,pt,view=[0..10,-3..3]);**
> **print(`Speed in Red, curvature in Green, torsion in Blue`);**



Speed in Red, curvature in Green, torsion in Blue

Part V: TNB Frame - Computation and Visualization

The steps here resemble the ones above, except that we choose a simpler function and extend the formulas to the unit normal and binormal vectors.

```
> restart:
  with(linalg):
  with(plots):
```

Warning, the protected names norm and trace have been redefined and unprotected

Warning, the name changecoords has been redefined

```
> mag:=proc(v) RETURN(sqrt(multiply(v,v))): end :
  r:=[sin(2*t),cos(2*t),t^2/10]:
  v:=diff(r,t):
  a:=diff(v,t):
  speed:=simplify(mag(v)):
  utan:=simplify(expand(v/speed)):
  curvature:=simplify(mag(linalg[crossprod](v,a))/speed^3):
  torsion:=simplify(det(matrix(3,3,[v,a,diff(a,t)]))/simplify(multiply(crossprod(v,a),crossprod
  (v,a)))):
  top:=diff(utan,t):
  bottom:=simplify((mag(top))):
  un:=simplify(expand(simplify(top/bottom))):
  ubn:=convert(simplify(crossprod(utan,un)),list):
  print(`The position vector is `, r):
  print(`The velocity vector is `, v):
  print(`The acceleration vector is `, a):
```

```
print(`The speed is `, speed):
print(`The unit tangent vector is `, utan):
print(`The curvature is `, curvature):
print(`The torsion is `, torsion):
print(`The unit normal is `,un):
print(`The unit binormal is `,ubn):
```

$$\textit{The position vector is}\ ,\left[\sin(2\,t),\cos(2\,t),\frac{t^2}{10}\right]$$

$$\textit{The velocity vector is}\ ,\left[2\cos(2\,t),-2\sin(2\,t),\frac{t}{5}\right]$$

$$\textit{The acceleration vector is}\ ,\left[-4\sin(2\,t),-4\cos(2\,t),\frac{1}{5}\right]$$

$$\textit{The speed is}\ ,\frac{\sqrt{t^2+100}}{5}$$

$$\textit{The unit tangent vector is}\ ,\left[\frac{10\,(2\cos(t)^2-1)}{\sqrt{t^2+100}},-\frac{20\sin(t)\cos(t)}{\sqrt{t^2+100}},\frac{t}{\sqrt{t^2+100}}\right]$$

$$\textit{The curvature is}\ ,\frac{50\sqrt{4\,t^2+401}}{(t^2+100)^{\left(\frac{3}{2}\right)}}$$

$$\textit{The torsion is}\ ,-\frac{40\,t}{4\,t^2+401}$$

$$\textit{The unit normal is}\ ,\left[-\frac{4\sin(t)\cos(t)\,t^2+400\sin(t)\cos(t)+2\,t\cos(t)^2-t}{(t^2+100)^{\left(\frac{3}{2}\right)}\sqrt{\frac{4\,t^2+401}{(t^2+100)^2}}},\right.$$

$$-\frac{2\left(-\sin(t)\cos(t)\,t+2t^2\cos(t)^2+200\cos(t)^2-t^2-100\right)}{(t^2+100)^{\left(\frac{3}{2}\right)}\sqrt{\frac{4t^2+401}{(t^2+100)^2}}}, \frac{10}{(t^2+100)^{\left(\frac{3}{2}\right)}\sqrt{\frac{4t^2+401}{(t^2+100)^2}}}$$

The unit binormal is ,

$$\left[\frac{2\left(2t\cos(t)^2-t-\sin(t)\cos(t)\right)}{\sqrt{\frac{4t^2+401}{(t^2+100)^2}}\,(t^2+100)}, -\frac{4\sin(t)\cos(t)\,t+2\cos(t)^2-1}{\sqrt{\frac{4t^2+401}{(t^2+100)^2}}\,(t^2+100)}, -\frac{20}{\sqrt{\frac{4t^2+401}{(t^2+100)^2}}\,(t^2\right.$$

Let's find the tangential and normal components of the accelerations.

```
> at:=combine(dotprod(a,utan)):
  an:=radsimp(simplify(dotprod(a,un))):
  print(`The tangential component of acceleration is `,at);
  print(`The normal component of acceleration is `,an);
```

The tangential component of acceleration is ,

$$-40\sin(2t)\left(\frac{\cos(2t)}{\sqrt{t^2+100}}\right)+40\cos(2t)\left(\frac{\sin(2t)}{\sqrt{t^2+100}}\right)+\frac{1}{5}\left(\frac{t}{\sqrt{t^2+100}}\right)$$

The normal component of acceleration is , $4\sin(2t)\left(\frac{4\sin(t)\cos(t)\,t^2+400\sin(t)\cos(t)+2t\,c\right.$ $\left.\overline{\sqrt{t^2+100}\sqrt{4t^2+401}}\right.$

$$-8\cos(2t)\left(\frac{\sin(t)\cos(t)\,t-2t^2\cos(t)^2-200\cos(t)^2+t^2+100}{\sqrt{t^2+100}\sqrt{4t^2+401}}\right)+2\left(\frac{1}{\sqrt{t^2+100}\sqrt{4t^2+}}\right.$$

You can get a visual perspective of the tangential and normal components of the acceleration in two dimensions by exploring the Java applet, "Tangent and Normal Vectors."

If you compare these components found by the dot products to the formulas given in your book, you will see that they agree, once the absolute value sign is removed from the curvature formula.

```
> print(`tangential component of acceleration is`,diff(speed,t));
  print(`normal component of acceleration is`,curvature*speed^2);
```

$$\textit{tangential component of acceleration is, } \frac{t}{5\sqrt{t^2+100}}$$

$$\textit{normal component of acceleration is, } \frac{2\sqrt{4t^2+401}}{\sqrt{t^2+100}}$$

The equality of the tangential components can be verified easily if you notice that the first two terms of the result found previously by dotting the acceleration vector into the unit tangent vector subtract out with one another. The equality of the normal components is not as obvious. You can see the advantage in using the formulas used in the latter part, due to the complexity of the unit normal vector. To see the results in more simplified form, check out the *Mathematica* version on your CD.

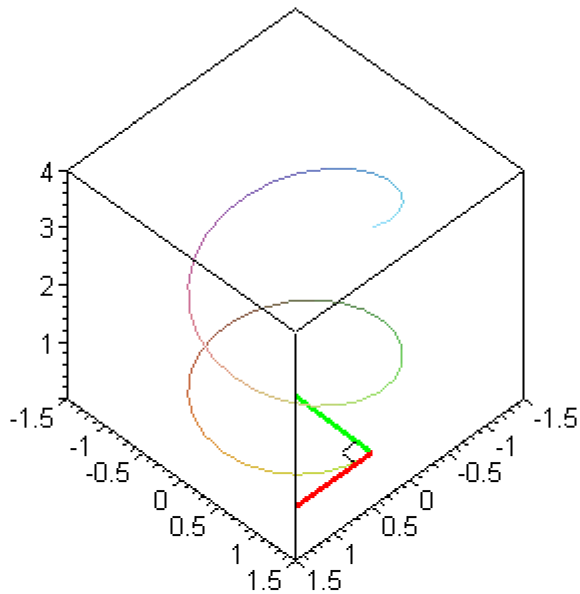
The following set of commands will plot the unit tangent, unit normal, and unit binormal vectors as you move along the curve.

After the plots are generated, click anywhere on the graph to bring up the animation toolbar. From the toolbar, click on **Play** (large arrow) to view the animation and click on either of the **double arrow buttons** to slow down or speed up the animation.

Remember, you can click and drag the plot to view it from different angles.

```
> with(plottools):temp:=1:
  p1:=spacecurve(r,t=0..2*Pi,view=[-1.5..1.5,-1.5..1.5,0..4],axes=boxed):
  for i from 0 to evalf(2*Pi) by evalf(Pi/4) do
   p1:=spacecurve(r,t=0..2*Pi,view=[-1.5..1.5,-1.5..1.5,0..4],axes=boxed):
   p2:=plottools[line](evalf(subs(t=i,r)),evalf(subs(t=i,r+utan)),color=red,thickness=3):
   p3:=plottools[line](evalf(subs(t=i,r)),evalf(subs(t=i,r+un)),color=green,thickness=3):
   p4:=plottools[line](evalf(subs(t=i,r)),subs(t=i,r+ubn),color=blue,thickness=3):
   p5:=display(line(evalf(subs(t=i,r+0.2*utan)),evalf(subs(t=i,r+0.2*utan+0.2*ubn))),
   line(evalf(subs(t=i,r+0.2*ubn)),evalf(subs(t=i,r+0.2*utan+0.2*ubn))),
   line(evalf(subs(t=i,r+0.2*utan)),evalf(subs(t=i,r+0.2*utan+0.2*un))),
   line(evalf(subs(t=i,r+0.2*un)),evalf(subs(t=i,r+0.2*utan+0.2*un))),
   line(evalf(subs(t=i,r+0.2*un)),evalf(subs(t=i,r+0.2*un+0.2*ubn))),
   line(evalf(subs(t=i,r+0.2*ubn)),evalf(subs(t=i,r+0.2*un+0.2*ubn))),
   line(evalf(subs(t=i,r+0.2*utan+0.2*ubn)),evalf(subs(t=i,r+0.2*utan+0.2*ubn+0.2*un))),
   line(evalf(subs(t=i,r+0.2*utan+0.2*un)),evalf(subs(t=i,r+0.2*utan+0.2*ubn+0.2*un))), line
  (evalf(subs(t=i,r+0.2*ubn+0.2*un)),evalf(subs(t=i,r+0.2*utan+0.2*ubn+0.2*un))),color=bla
   P[temp]:=display(p1,p2,p3,p4,p5):
   temp:=temp+1:
  od:
  P:=convert(P,list):
  display(P,insequence=true);
  print(`Tangent in RED, unit normal in GREEN, unit binormal in BLUE.`);
```

Warning, the assigned name arrow now has a global binding



Tangent in RED, unit normal in GREEN, unit binormal in BLUE.

You Try It: Part V

The computational code is written so that all you have to do is change the initial vector (**r**) function and then re-execute the commands. We suggest that you choose some of the homework problems from your text for the TNB frame, since the computation for the unit normal and unit binormal can get you very bogged down, even in *Maple*. You may also wish to change the domain over which you extend your function in the second section.

If you check out the following example, expect to wait a few minutes for the computations.

```
> restart:
  with(plots):
  with(linalg):
```

Warning, the name changecoords has been redefined

Warning, the protected names norm and trace have been redefined and unprotected

```
> mag:=proc(v) RETURN(sqrt(multiply(v,v))): end :
```

```
> r:=[2*t,cos(t-1),exp(-t)]:
```

```
> v:=diff(r,t):
```

```
> a:=diff(v,t):
```

```
> speed:=simplify(mag(v)):

> utan:=simplify(v/speed):

> curvature:=simplify(mag(linalg[crossprod](v,a))/speed^3):

> torsion:=simplify(det(matrix(3,3,[v,a,diff(a,t)]))/simplify(multiply(crossprod(v,a),crossprod
(v,a)))):

> top:=diff(utan,t):

> bottom:=simplify((mag(top))):

> un:=simplify(expand(simplify(top/bottom))):

> ubn:=convert(simplify(crossprod(utan,un)),list):

> print(`The position vector is `, r):

> print(`The velocity vector is `, v):

> print(`The acceleration vector is `, a):

> print(`The speed is `, speed):

> print(`The unit tangent vector is `, utan):

> print(`The curvature is `, curvature):

> print(`The torsion is `, torsion):

> print(`The unit normal is `,un):

> print(`The unit binormal is `,ubn):
```

$$\textit{The position vector is}\ ,\ [2\,t,\ \cos(t-1),\ \mathbf{e}^{(-t)}]$$

The velocity vector is , $[2, -\sin(t-1), -\mathbf{e}^{(-t)}]$

The acceleration vector is , $[0, -\cos(t-1), \mathbf{e}^{(-t)}]$

The speed is , $\sqrt{5-\cos(t-1)^2+\mathbf{e}^{(-2\,t)}}$

The unit tangent vector is , $\dfrac{[2, -\sin(t-1), -\mathbf{e}^{(-t)}]}{\sqrt{5-\cos(t-1)^2+\mathbf{e}^{(-2\,t)}}}$

The curvature is , $\dfrac{\sqrt{5\,\mathbf{e}^{(-2\,t)}+2\sin(t-1)\,\mathbf{e}^{(-2\,t)}\cos(t-1)+4\cos(t-1)^2}}{(5-\cos(t-1)^2+\mathbf{e}^{(-2\,t)})^{\left(\frac{3}{2}\right)}}$

The torsion is , $-\dfrac{2\,\mathbf{e}^{(-t)}(-\cos(t-1)+\sin(t-1))}{5\,\mathbf{e}^{(-2\,t)}+2\sin(t-1)\,\mathbf{e}^{(-2\,t)}\cos(t-1)+4\cos(t-1)^2}$

The unit normal is , $\Big[-2\,($

$1-2\,\mathbf{e}^{(2\,t)}\sin(t)\cos(1)^2\cos(t)-\mathbf{e}^{(2\,t)}\sin(1)\cos(1)+2\,\mathbf{e}^{(2\,t)}\cos(t)^2\sin(1)\cos(1)+\mathbf{e}^{(2\,t)}$

$)\Big/\Big((-4\,\mathbf{e}^{(2\,t)}+2\cos(t)^2\cos(1)^2\,\mathbf{e}^{(2\,t)}+2\cos(t)\cos(1)\sin(t)\sin(1)\,\mathbf{e}^{(2\,t)}-\cos(1)^2\,\mathbf{e}^{(2}$

$-\mathbf{e}^{(2\,t)}\cos(t)^2-1)(-(-4\,\mathbf{e}^{(2\,t)}+2\cos(t)^2\cos(1)^2\,\mathbf{e}^{(2\,t)}+2\cos(t)\cos(1)\sin(t)\sin(1)\,\mathbf{e}^{(2}$

$-\cos(1)^2\,\mathbf{e}^{(2\,t)}-\mathbf{e}^{(2\,t)}\cos(t)^2-1)\,\mathbf{e}^{(-2\,t)})^{\left(\frac{1}{2}\right)}\Big((\mathbf{e}^{t})^2\Big(5+4\sin(t)\cos(1)^2\cos(t)+2\sin($

$-4\cos(t)^2\sin(1)\cos(1)-2\cos(t)\sin(t)+8\cos(t)^2\cos(1)^2(\mathbf{e}^{t})^2+8\cos(t)\cos(1)\sin(t)\sin($

$+4\,(\mathbf{e}^{t})^2-4\cos(1)^2(\mathbf{e}^{t})^2-4\,(\mathbf{e}^{t})^2\cos(t)^2\Big)\Big/\Big(1-4\cos(t)\cos(1)\sin(t)\sin(1)\,\mathbf{e}^{(2\,t)}$

$-4\cos(t)\cos(1)^3\sin(t)\sin(1)\,\mathbf{e}^{(4\,t)}-4\cos(t)^3\cos(1)\sin(t)\sin(1)\,\mathbf{e}^{(4\,t)}-8\cos(t)^2\cos(1)^4$

$$-8\cos(t)^4\cos(1)^2 e^{(4t)} + 8\cos(1)^2 e^{(4t)} - 10\cos(t)^2\cos(1)^2 e^{(4t)} + 8\cos(t)^4\cos(1)^4$$

$$+\cos(1)^4 e^{(4t)} + 2e^{(2t)}\cos(t)^2 + 8e^{(2t)} - 4\cos(t)^2\cos(1)^2 e^{(2t)} + 2\cos(1)^2 e^{(2t)} + 1$$

$$-16\cos(t)\cos(1)\sin(t)\sin(1)e^{(4t)} + 8\cos(t)^3\cos(1)^3\sin(t)\sin(1)e^{(4t)} + 8e^{(4t)}\cos(t)^2$$

$$\left.\left.+e^{(4t)}\cos(t)^4\right)^{\left(\frac{1}{2}\right)}\right),\ ($$

$$\sin(t)\cos(1) - \cos(t)\sin(1) + 4\cos(t)\cos(1)e^{(2t)} + \cos(t)\cos(1) + \sin(t)\sin(1) + 4\sin(t)\sin($$

$$\Big/ \left((-4e^{(2t)} + 2\cos(t)^2\cos(1)^2 e^{(2t)} + 2\cos(t)\cos(1)\sin(t)\sin(1)e^{(2t)} - \cos(1)^2 e^{(2}\right.$$

$$-e^{(2t)}\cos(t)^2 - 1)\,(-(-4e^{(2t)} + 2\cos(t)^2\cos(1)^2 e^{(2t)} + 2\cos(t)\cos(1)\sin(t)\sin(1)e^{(2}$$

$$-\cos(1)^2 e^{(2t)} - e^{(2t)}\cos(t)^2 - 1)\,e^{(-2t)})^{\left(\frac{1}{2}\right)} \left((e^t)^2\left(5 + 4\sin(t)\cos(1)^2\cos(t) + 2\sin(\right.\right.$$

$$-4\cos(t)^2\sin(1)\cos(1) - 2\cos(t)\sin(t) + 8\cos(t)^2\cos(1)^2(e^t)^2 + 8\cos(t)\cos(1)\sin(t)\sin($$

$$\left.+4(e^t)^2 - 4\cos(1)^2(e^t)^2 - 4(e^t)^2\cos(t)^2\right) \Big/ \left(1 - 4\cos(t)\cos(1)\sin(t)\sin(1)e^{(2t)}\right.$$

$$-4\cos(t)\cos(1)^3\sin(t)\sin(1)e^{(4t)} - 4\cos(t)^3\cos(1)\sin(t)\sin(1)e^{(4t)} - 8\cos(t)^2\cos(1)^4$$

$$-8\cos(t)^4\cos(1)^2 e^{(4t)} + 8\cos(1)^2 e^{(4t)} - 10\cos(t)^2\cos(1)^2 e^{(4t)} + 8\cos(t)^4\cos(1)^4$$

$$+\cos(1)^4 e^{(4t)} + 2e^{(2t)}\cos(t)^2 + 8e^{(2t)} - 4\cos(t)^2\cos(1)^2 e^{(2t)} + 2\cos(1)^2 e^{(2t)} + 1$$

$$-16\cos(t)\cos(1)\sin(t)\sin(1)e^{(4t)} + 8\cos(t)^3\cos(1)^3\sin(t)\sin(1)e^{(4t)} + 8e^{(4t)}\cos(t)^2$$

$$\left.\left.+e^{(4t)}\cos(t)^4\right)^{\left(\frac{1}{2}\right)}\right),\ (-2\sin(t)\cos(1)^2\cos(t) - \sin(1)\cos(1) + 2\cos(t)^2\sin(1)\cos(1) + \mathrm{co}$$

$$+2\cos(t)^2\cos(1)^2 + 2\cos(t)\cos(1)\sin(t)\sin(1) - 4 - \cos(1)^2 - \cos(t)^2)\,e^t \Big/ \left((-4e^{(2t)}\right.$$

$$+2\cos(t)^2\cos(1)^2 e^{(2t)} + 2\cos(t)\cos(1)\sin(t)\sin(1)e^{(2t)} - \cos(1)^2 e^{(2t)} - e^{(2t)}\cos(t$$

$$-4e^{(2t)} + 2\cos(t)^2\cos(1)^2 e^{(2t)} + 2\cos(t)\cos(1)\sin(t)\sin(1)e^{(2t)} - \cos(1)^2 e^{(2t)} - e^{(2}$$

$$- 1)\, e^{(-2t)})^{\left(\frac{1}{2}\right)} \Big((e^t)^2 \Big(5 + 4\sin(t)\cos(1)^2\cos(t) + 2\sin(1)\cos(1) - 4\cos(t)^2\sin(1)\cos(1)$$

$$- 2\cos(t)\sin(t) + 8\cos(t)^2\cos(1)^2(e^t)^2 + 8\cos(t)\cos(1)\sin(t)\sin(1)(e^t)^2 + 4(e^t)^2 - 4\,\text{cc}$$

$$- 4(e^t)^2\cos(t)^2\Big) \Big/ (1 - 4\cos(t)\cos(1)\sin(t)\sin(1)\, e^{(2t)} - 4\cos(t)\cos(1)^3\sin(t)\sin(1)$$

$$- 4\cos(t)^3\cos(1)\sin(t)\sin(1)\, e^{(4t)} - 8\cos(t)^2\cos(1)^4 e^{(4t)} - 8\cos(t)^4\cos(1)^2 e^{(4t)}$$

$$+ 8\cos(1)^2 e^{(4t)} - 10\cos(t)^2\cos(1)^2 e^{(4t)} + 8\cos(t)^4\cos(1)^4 e^{(4t)} + \cos(1)^4 e^{(4t)}$$

$$+ 2e^{(2t)}\cos(t)^2 + 8e^{(2t)} - 4\cos(t)^2\cos(1)^2 e^{(2t)} + 2\cos(1)^2 e^{(2t)} + 16e^{(4t)}$$

$$- 16\cos(t)\cos(1)\sin(t)\sin(1)\, e^{(4t)} + 8\cos(t)^3\cos(1)^3\sin(t)\sin(1)\, e^{(4t)} + 8e^{(4t)}\cos(t)^2$$

$$+ e^{(4t)}\cos(t)^4)^{\left(\frac{1}{2}\right)}\Big)\Big]$$

The unit binormal is ,

$$\Big[- (-2\sin(t-1)\, e^t\sin(t)\cos(1)^2\cos(t) - \sin(t-1)\, e^t\sin(1)\cos(1)$$

$$+ 2\sin(t-1)\, e^t\cos(t)^2\sin(1)\cos(1) + \sin(t-1)\, e^t\cos(t)\sin(t) + 2\sin(t-1)\, e^t\cos(t)^2\cos(1$$

$$+ 2\sin(t-1)\, e^t\cos(t)\cos(1)\sin(t)\sin(1) - 4\sin(t-1)\, e^t - \sin(t-1)\, e^t\cos(1)^2 - \sin(t-1)\, e^t$$

$$- e^{(-t)}\sin(t)\cos(1) + e^{(-t)}\cos(t)\sin(1) - 4e^t\cos(t)\cos(1) - e^{(-t)}\cos(t)\cos(1) - e^{(-t)}\,\text{sin}$$

$$- 4e^t\sin(t)\sin(1)) \Big/ \Big(\sqrt{5 - \cos(t-1)^2 + e^{(-2t)}}\,(-4e^{(2t)} + 2\cos(t)^2\cos(1)^2 e^{(2t)}$$

$$+ 2\cos(t)\cos(1)\sin(t)\sin(1)\, e^{(2t)} - \cos(1)^2 e^{(2t)} - e^{(2t)}\cos(t)^2 - 1)(-(-4e^{(2t)}$$

$$+ 2\cos(t)^2\cos(1)^2 e^{(2t)} + 2\cos(t)\cos(1)\sin(t)\sin(1)\, e^{(2t)} - \cos(1)^2 e^{(2t)} - e^{(2t)}\cos(t$$

$$e^{(-2t)})^{\left(\frac{1}{2}\right)} \Big((e^t)^2 \Big(5 + 4\sin(t)\cos(1)^2\cos(t) + 2\sin(1)\cos(1) - 4\cos(t)^2\sin(1)\cos(1) - 2\,\text{c}$$

$$+ 8\cos(t)^2\cos(1)^2(e^t)^2 + 8\cos(t)\cos(1)\sin(t)\sin(1)(e^t)^2 + 4(e^t)^2 - 4\cos(1)^2(e^t)^2$$

$$- 4(e^t)^2\cos(t)^2\Big) \Big/ (1 - 4\cos(t)\cos(1)\sin(t)\sin(1)\, e^{(2t)} - 4\cos(t)\cos(1)^3\sin(t)\sin(1)$$

$$- 4\cos(t)^3\cos(1)\sin(t)\sin(1)\, e^{(4t)} - 8\cos(t)^2\cos(1)^4 e^{(4t)} - 8\cos(t)^4\cos(1)^2 e^{(4t)}$$

$$+8\cos(1)^2 e^{(4t)} - 10\cos(t)^2\cos(1)^2 e^{(4t)} + 8\cos(t)^4\cos(1)^4 e^{(4t)} + \cos(1)^4 e^{(4t)}$$
$$+2 e^{(2t)}\cos(t)^2 + 8 e^{(2t)} - 4\cos(t)^2\cos(1)^2 e^{(2t)} + 2\cos(1)^2 e^{(2t)} + 16 e^{(4t)}$$
$$-16\cos(t)\cos(1)\sin(t)\sin(1) e^{(4t)} + 8\cos(t)^3\cos(1)^3\sin(t)\sin(1) e^{(4t)} + 8 e^{(4t)}\cos(t)^2$$
$$\left.\left. + e^{(4t)}\cos(t)^4\right)^{\left(\frac{1}{2}\right)}\right), -2$$

$$(-e^{(-t)} + 2e^t\cos(t)^2\cos(1)^2 + 2e^t\cos(t)\cos(1)\sin(t)\sin(1) - 4e^t - e^t\cos(1)^2 - e^t\cos(t)^2$$
$$\sqrt{5-\cos(t-1)^2 + e^{(-2t)}}\,(-4e^{(2t)} + 2\cos(t)^2\cos(1)^2 e^{(2t)} + 2\cos(t)\cos(1)\sin(t)\sin(1)$$
$$-\cos(1)^2 e^{(2t)} - e^{(2t)}\cos(t)^2 - 1)\,(-(-4e^{(2t)} + 2\cos(t)^2\cos(1)^2 e^{(2t)}$$
$$+2\cos(t)\cos(1)\sin(t)\sin(1) e^{(2t)} - \cos(1)^2 e^{(2t)} - e^{(2t)}\cos(t)^2 - 1)\,e^{(-2t)})^{\left(\frac{1}{2}\right)}\left((e^t)^2\right.$$
$$+4\sin(t)\cos(1)^2\cos(t) + 2\sin(1)\cos(1) - 4\cos(t)^2\sin(1)\cos(1) - 2\cos(t)\sin(t)$$
$$+8\cos(t)^2\cos(1)^2(e^t)^2 + 8\cos(t)\cos(1)\sin(t)\sin(1)(e^t)^2 + 4(e^t)^2 - 4\cos(1)^2(e^t)^2$$
$$\left.-4(e^t)^2\cos(t)^2\right) \Big/ (1 - 4\cos(t)\cos(1)\sin(t)\sin(1) e^{(2t)} - 4\cos(t)\cos(1)^3\sin(t)\sin(1)$$
$$-4\cos(t)^3\cos(1)\sin(t)\sin(1) e^{(4t)} - 8\cos(t)^2\cos(1)^4 e^{(4t)} - 8\cos(t)^4\cos(1)^2 e^{(4t)}$$
$$+8\cos(1)^2 e^{(4t)} - 10\cos(t)^2\cos(1)^2 e^{(4t)} + 8\cos(t)^4\cos(1)^4 e^{(4t)} + \cos(1)^4 e^{(4t)}$$
$$+2 e^{(2t)}\cos(t)^2 + 8 e^{(2t)} - 4\cos(t)^2\cos(1)^2 e^{(2t)} + 2\cos(1)^2 e^{(2t)} + 16 e^{(4t)}$$
$$-16\cos(t)\cos(1)\sin(t)\sin(1) e^{(4t)} + 8\cos(t)^3\cos(1)^3\sin(t)\sin(1) e^{(4t)} + 8 e^{(4t)}\cos(t)^2$$
$$\left.\left. + e^{(4t)}\cos(t)^4\right)^{\left(\frac{1}{2}\right)}\right), -2\,(-\sin(t)\cos(1) + \cos(t)\sin(1) - 4\cos(t)\cos(1) e^{(2t)} - \cos(t)\cos$$
$$-\sin(t)\sin(1) - 4\sin(t)\sin(1) e^{(2t)} + \sin(t-1) - 2\sin(t-1) e^{(2t)}\sin(t)\cos(1)^2\cos(t)$$
$$-\sin(t-1) e^{(2t)}\sin(1)\cos(1) + 2\sin(t-1) e^{(2t)}\cos(t)^2\sin(1)\cos(1) + \sin(t-1) e^{(2t)}\cos$$
$$\Big/ \left(\sqrt{5-\cos(t-1)^2 + e^{(-2t)}}\,(-4e^{(2t)} + 2\cos(t)^2\cos(1)^2 e^{(2t)} + 2\cos(t)\cos(1)\sin(t)\right.$$

$$- \cos(1)^2 e^{(2t)} - e^{(2t)} \cos(t)^2 - 1) (-(-4 e^{(2t)} + 2 \cos(t)^2 \cos(1)^2 e^{(2t)}$$

$$+ 2 \cos(t) \cos(1) \sin(t) \sin(1) e^{(2t)} - \cos(1)^2 e^{(2t)} - e^{(2t)} \cos(t)^2 - 1) e^{(-2t)})^{\left(\frac{1}{2}\right)} \Big((e^t)^2$$

$$+ 4 \sin(t) \cos(1)^2 \cos(t) + 2 \sin(1) \cos(1) - 4 \cos(t)^2 \sin(1) \cos(1) - 2 \cos(t) \sin(t)$$

$$+ 8 \cos(t)^2 \cos(1)^2 (e^t)^2 + 8 \cos(t) \cos(1) \sin(t) \sin(1) (e^t)^2 + 4 (e^t)^2 - 4 \cos(1)^2 (e^t)^2$$

$$- 4 (e^t)^2 \cos(t)^2 \Big) \Big/ (1 - 4 \cos(t) \cos(1) \sin(t) \sin(1) e^{(2t)} - 4 \cos(t) \cos(1)^3 \sin(t) \sin(1) e \cdots$$

$$- 4 \cos(t)^3 \cos(1) \sin(t) \sin(1) e^{(4t)} - 8 \cos(t)^2 \cos(1)^4 e^{(4t)} - 8 \cos(t)^4 \cos(1)^2 e^{(4t)}$$

$$+ 8 \cos(1)^2 e^{(4t)} - 10 \cos(t)^2 \cos(1)^2 e^{(4t)} + 8 \cos(t)^4 \cos(1)^4 e^{(4t)} + \cos(1)^4 e^{(4t)}$$

$$+ 2 e^{(2t)} \cos(t)^2 + 8 e^{(2t)} - 4 \cos(t)^2 \cos(1)^2 e^{(2t)} + 2 \cos(1)^2 e^{(2t)} + 16 e^{(4t)}$$

$$- 16 \cos(t) \cos(1) \sin(t) \sin(1) e^{(4t)} + 8 \cos(t)^3 \cos(1)^3 \sin(t) \sin(1) e^{(4t)} + 8 e^{(4t)} \cos(t)^2$$

$$+ e^{(4t)} \cos(t)^4))^{\left(\frac{1}{2}\right)})]$$

The following set of commands will plot the unit tangent, unit normal, and unit binormal vectors as you move along the curve.

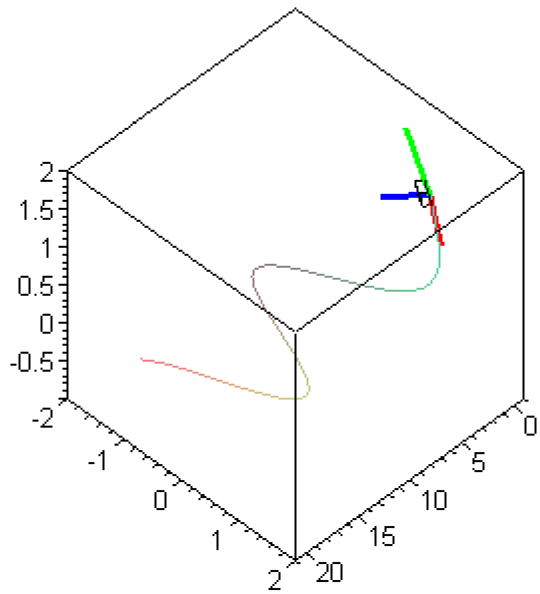
After the plots are generated, click anywhere on the graph to bring up the animation toolbar. From the toolbar, click on **Play** (large arrow) to view the animation and lcik on either of the **double arrow buttons** to slow down or speed up the animation.

Remember, you can click and drag the plot to view it from different angles.

```
> with(plottools):temp:=1:

> for i from 0 to 10 by 1 do

>  p1:=spacecurve(r,t=0..10,view=[-1..21,-2..2,-1..2],axes=boxed,scaling=constrained):

>  p2:=plottools[line](evalf(subs(t=i,r)),evalf(subs(t=i,r+utan)),color=red,thickness=3):

>  p3:=plottools[line](evalf(subs(t=i,r)),evalf(subs(t=i,r+un)),color=green,thickness=3):

>  p4:=plottools[line](evalf(subs(t=i,r)),subs(t=i,r+ubn),color=blue,thickness=3):
```

```
> p5:=display(line(evalf(subs(t=i,r+0.2*utan)),evalf(subs(t=i,r+0.2*utan+0.2*ubn))),
> line(evalf(subs(t=i,r+0.2*ubn)),evalf(subs(t=i,r+0.2*utan+0.2*ubn))),
> line(evalf(subs(t=i,r+0.2*utan)),evalf(subs(t=i,r+0.2*utan+0.2*un))),
> line(evalf(subs(t=i,r+0.2*un)),evalf(subs(t=i,r+0.2*utan+0.2*un))),
> line(evalf(subs(t=i,r+0.2*un)),evalf(subs(t=i,r+0.2*un+0.2*ubn))),
> line(evalf(subs(t=i,r+0.2*ubn)),evalf(subs(t=i,r+0.2*un+0.2*ubn))),
> line(evalf(subs(t=i,r+0.2*utan+0.2*ubn)),evalf(subs(t=i,r+0.2*utan+0.2*ubn+0.2*un))),
> line(evalf(subs(t=i,r+0.2*utan+0.2*un)),evalf(subs(t=i,r+0.2*utan+0.2*ubn+0.2*un))),
> line(evalf(subs(t=i,r+0.2*ubn+0.2*un)),evalf(subs
  (t=i,r+0.2*utan+0.2*ubn+0.2*un))),color=black):
> P[temp]:=display(p1,p2,p3,p4,p5):
> temp:=temp+1:
> od:
> P:=convert(P,list):
> display(P,insequence=true);
  print(`Tangent in RED, unit normal in GREEN, unit binormal in BLUE.`);
```

Warning, the assigned name arrow now has a global binding



Tangent in RED, unit normal in GREEN, unit binormal in BLUE.

>