Using Riemann Sums to Estimate Areas, Volumes, and Lengths of Arc

Introduction

OBJECTIVE: Use Riemann sums to approximate areas, volumes, and lengths of arc. Construct accumulation functions, and see how the accumulated quantities converge on the antiderivative in the limit.

In this module, we explore the use of Riemann sums to estimate areas, volumes of revolution, and the length of a curve. We first write an expression for ΔQ_i, where *Q* is the quantity of interest (area, volume, or arc length); then we look at $\frac{\Delta Q_i}{\Delta x}$ as the rate at which *Q* accumulates with respect to increments in *x*; we form a list of accumulated quantities, Q_i; and finally, we use the list of Q_i's to reconstruct $\frac{\Delta Q_i}{\Delta x}$.

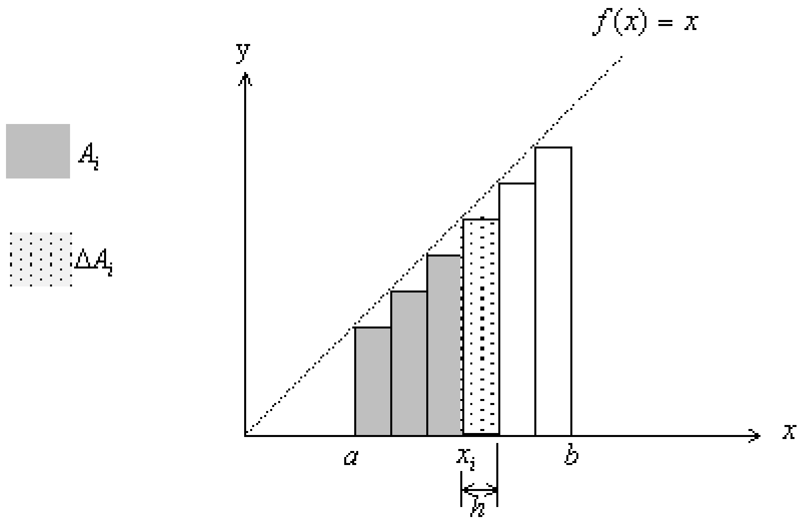
Technology Guidelines

NOTE: If you have just finished a module, restart *Mathematica* or close the *Kernel* before executing a new module.
TO OPEN CELLS, put your cursor on the right cell bracket and double click.
INITIALIZATION CELLS
When asked if you want to ". . . automatically evaluate all the initialization cells in the notebook . . . ," respond by pressing the "Yes" button.
TO STOP AN EXECUTION
Select the *Kernel* pull-down menu and click on *Abort Evaluation.*
ORDER OF EXECUTION
Execute cells in the order given. Do not skip any Input cells within a given notebook.
SAVING NOTEBOOKS
You can save anytime to any directory you choose, and it is wise to save often. However, before you do your final save, delete all your output by selecting the *Delete All Output* selection under the *Kernel* pull-down menu.
EXPERIENCING MAJOR PROBLEMS

Save if appropriate, then shut down *Mathematica* and start it up again.

Part I: Areas

Chapter 5, Sections 3 and 4



Note: In this exercise, ΔA_i represents the area of the i^{th} rectangle, as shown in the figure above.

In this Part, we will do the following:
1. Form a function, ΔA(i), that gives the area of the i^{th} rectangle in terms of f(x_i) and *h*.
2. Form a rate-of-change function, $\frac{\Delta A(i)}{\Delta x}$, that gives the rate of accumulation of area at x=a+i h.
3. Form a function, **areaSum[k_]**, that gives the sum of the areas of the rectangles, that is, the ΔA(i)'s, as *i* ranges from 0 to *k*. This approximates the area under the graph of y=f(x), between x=a and x=a+k h.
4. Form a function that gives the slopes of the secant lines on the graph of the A(k) function found in step 3, and show that this is the same as the rate-of-change function found in step 2.
5. In the You Try It exercise, you explore what happens as the number of rectangles goes to infinity while *h* goes to 0.

Our first objective is to write ΔA_i , the area of the i^{th} rectangle, in terms of f(x_i) and *h* where x_i = a+i h. At first, we let a = 0, b = 5, and n=10.

In[17]:=

```
Clear[f, ΔA, a, b, h, n];

f[x_] = x;

a = 0;

b = 5;

n = 10;

h = (b - a) / n;

ΔA[i_] = f[a + i * h] * h
```

Out[23]=

$$\frac{i}{4}$$

Note that if f(x_i) is negative and Δ x=h is positive, then ΔA_i is negative. The areas can be positive, negative, or zero, and so we refer to them as "signed areas." Areas above the *x*-axis are positive, and areas below the *x*-axis are negative, provided Δ x=h is positive.

Next, we form the rate-of-change function, $\frac{\Delta A(i)}{\Delta x}$. The function $\frac{\Delta A(i)}{\Delta x}$ is the rate at which the total signed area of the rectangles accumulates with each increment of Δ x=h added to *x*. We generate a list of the rate-of-change values for each x_i =a+i h, plot the rate of change versus x_i , and then assign the graph to the symbol name **p0** for later reference.

In[24]:=

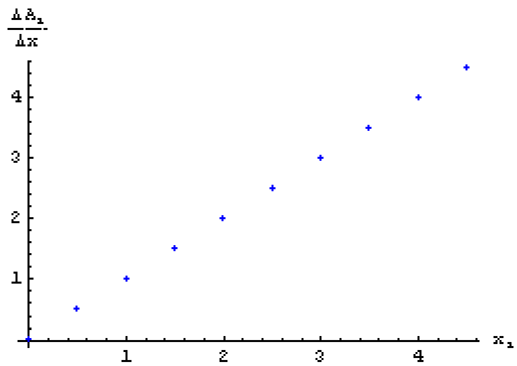
```
rateList = Table[{a + i h, ΔA[i]/h}, {i, 0, n - 1}]
```

Out[24]=

$$\left\{\{0, 0\}, \left\{\frac{1}{2}, \frac{1}{2}\right\}, \{1, 1\}, \left\{\frac{3}{2}, \frac{3}{2}\right\}, \{2, 2\}, \left\{\frac{5}{2}, \frac{5}{2}\right\}, \{3, 3\}, \left\{\frac{7}{2}, \frac{7}{2}\right\}, \{4, 4\}, \left\{\frac{9}{2}, \frac{9}{2}\right\}\right\}$$

In[25]:=

```
p0 = ListPlot[rateList,
  PlotStyle → {RGBColor[0, 0, 1], PointSize[0.
  AxesLabel → {"xi", "ΔAi/Δx"}];
```



You have probably noticed that, for the area problem, the rate at which the total area of the rectangles accumulates with each added increment *h* is simply f(x_i), the height of the i^{th} rectangle. That is, $\frac{\Delta A(i)}{\Delta x}$ =f(x_i).

Now we form a function that calculates the left Riemann sum of the ΔA_i's for the first *k* rectangles, where *k* can range between 0 and n.

In[26]:=

$$\texttt{areaSum[k\_] :=} \sum_{i=0}^{k-1} \texttt{ΔA[i];}$$

For example, we can use **areaSum[ ]** to calculate the total signed area of the first four rectangles.

In[27]:=

```
areaSum[4]
```

Out[27]=

$$\frac{3}{2}$$

Or we can calculate the area of all ten rectangles. (Recall that we set n=10.)

In[28]:=

```
areaSum[10]
```

Out[28]=

$$\frac{45}{4}$$

Next, we form a list of the ordered pairs, in which the first element of each ordered pair is the x-coordinate of the left side of the last rectangle in the sum, and the second element in each ordered pair is the area of the first k rectangles.

In[29]:=

```
arealist = Table[{a + k * h, areaSum[k]}, {k, 0,
```

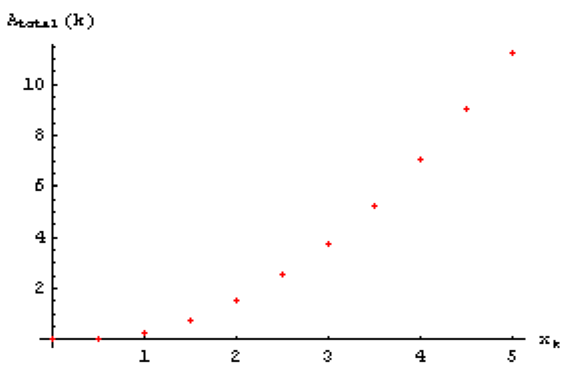
Out[29]=

$$\left\{\{0, 0\}, \left\{\frac{1}{2}, 0\right\}, \left\{1, \frac{1}{4}\right\}, \left\{\frac{3}{2}, \frac{3}{4}\right\}, \left\{2, \frac{3}{2}\right\}, \left\{\frac{5}{2}, \frac{5}{2}\right\}, \left\{3, \frac{15}{4}\right\}, \left\{\frac{7}{2}, \frac{21}{4}\right\}, \{4, 7\}, \left\{\frac{9}{2}, 9\right\}, \left\{5, \frac{45}{4}\right\}\right\}$$

Now we plot the list of points and assign it to the symbol name **p1** for later use.

In[30]:=

```
p1 = ListPlot[arealist,
  PlotStyle → {RGBColor[1, 0, 0], PointSize[0.
  AxesLabel → {"x_k", "A_total(k)"}];
```



The function **areaSum[k]** looks as though it might be a quadratic polynomial, so now we use the **Fit[ ]** command to see if there is a polynomial function that passes through the points.

In[31]:=

```
A_approx[x_] = Fit[arealist, {1, x, x^2}, x]
```

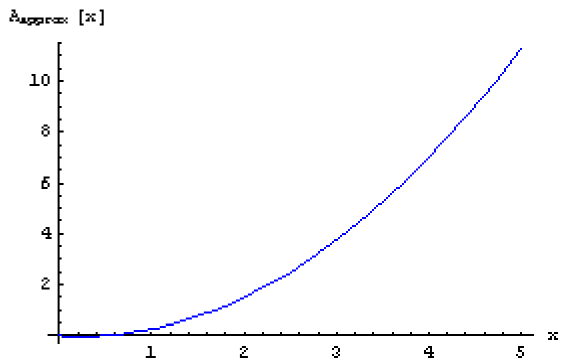
Out[31]=

$$-2.65704\times 10^{-16} - 0.25\,x + 0.5\,x^2$$

Now we plot $A_{approx}[x]$ and the points in **arealist** to show that the points do in fact lie on the graph of the quadratic polynomial. First, we plot $A_{approx}[x]$ and then show the two graphs on the same plot.

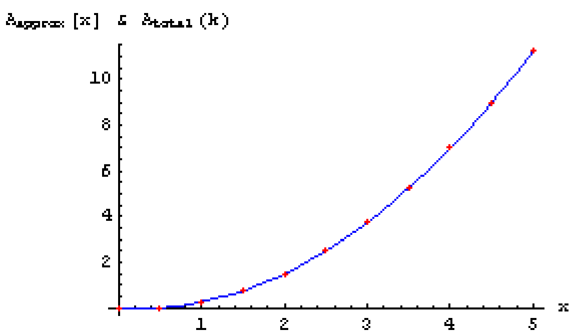
In[32]:=

```
p2 = Plot[Aapprox[x], {x, a, a + n h}, PlotStyle →
  AxesLabel → {"x", "Aapprox[x]"}];
```



In[33]:=

```
Show[{p2, p1}, AxesLabel → {"x", "Aapprox[x] &
```



The points appear to lie on the curve.

The slopes of the secant lines taken from the **areaSum[k_]** graph above give the rate at which the total area of the rectangles accumulates with each *h* increment of *x*. Now let's calculate the slopes of the secant lines between consecutive pairs of points on the graph above and plot them.

In[34]:=

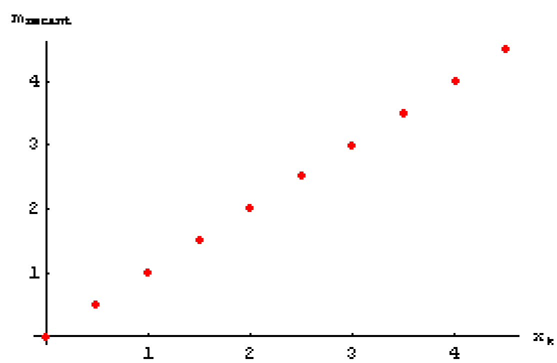
$$\texttt{msecant} = \texttt{Table}\left[\left\{a + k\,h,\ \frac{\texttt{areaSum[k + 1] - areaSu}}{h}\right.\right.$$

Out[34]=

$$\left\{\{0, 0\}, \left\{\frac{1}{2}, \frac{1}{2}\right\}, \{1, 1\}, \left\{\frac{3}{2}, \frac{3}{2}\right\}, \{2, 2\}, \left\{\frac{5}{2}, \frac{5}{2}\right\}, \{3, 3\}, \left\{\frac{7}{2}, \frac{7}{2}\right\}, \{4, 4\}, \left\{\frac{9}{2}, \frac{9}{2}\right\}\right\}$$

In[35]:=

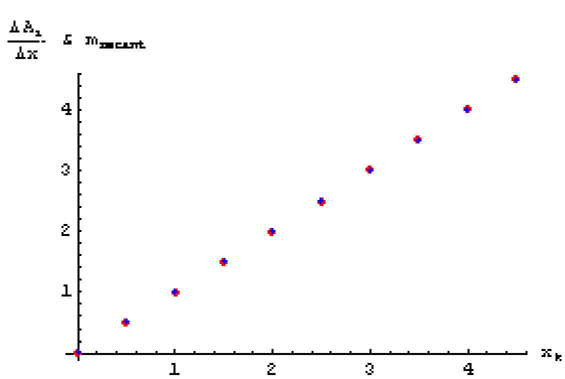
```
p4 = ListPlot[msecant,
  PlotStyle → {RGBColor[1, 0, 0], PointSize[0.
  AxesLabel → {"xk", "msecant"}];
```



But this is the same thing as $\frac{\Delta A(i)}{\Delta x}$ that we calculated at the beginning of this exercise. To confirm this, we show **msecants** and $\frac{\Delta A(i)}{\Delta x}$ together on the same graph.

In[36]:=

$$\texttt{Show}\left[\{\texttt{p4}, \texttt{p0}\}, \texttt{AxesLabel} \to \left\{\texttt{"}x_k\texttt{"}, \texttt{"}\frac{\Delta A_i}{\Delta x}\ \&\ m_{sec}\right.\right.$$



It looks as though we've come full circle. In the "You Try It" exercise that follows, you explore what happens as *n*, the number of rectangles, goes to infinity and as h→0.

You Try It: Part I - Taking it to the Limit and Other Functions

Chapter 5, Sections 3 and 4

To help you with these exercises, we copy all of the commands from Part I into a single cell, the one that follows. In each exercise, you are asked to change some of the red entries and answer the questions. (Note that we have hidden all but the input commands and put them in a command called **areas**, the last command in the following cell. In addition, we suppress displaying the lists of values that are generated because they become too large when *n* is a big number.)

1. For the function f(x)=x, considered in Part I, increase the number of rectangles, *n*, so that the total signed area of the rectangles approaches the total signed area under the graph of f(x). Try n=25, 50, 100, 250, 500. (For larger values of *n*, the evaluation of the commands will take a while because of the large number of ΔA_i 's that must be added to form each value of **areaSum[k]**. There are more efficient ways to calculate these sums, and these are investigated in the module "Riemann, Trapezoids, and Simpson," included in this supplement. As *n* gets larger, what function is **A[x]** approaching? Also, as *n* gets larger, Δ x=h gets closer to 0. What is the $\lim_{\Delta x \to 0} \frac{\Delta A_i}{\Delta x}$? In the limit as n→∞ and h→0, what is the relationship between $\frac{\Delta A_i}{\Delta x}$ and **A[x]**?

2. Repeat Exercise 1 for the following: f(x)=x^2 , a=-5, b=5, and **fitlist** ={1, x, x^2 , x^3 }.

3. Repeat Exercise 1 for the following: f(x)=cos x, a=0, b=2π, and **fitlist** = {1, Cos[x], Sin [x] }.

4. Repeat Exercise 1 for the following: f(x)=e^x , a=-1, b=1, and **fitlist** ={1, E^x }. (Note that in *Mathematica*, Euler's number *e* is represented by upper case *E*.)

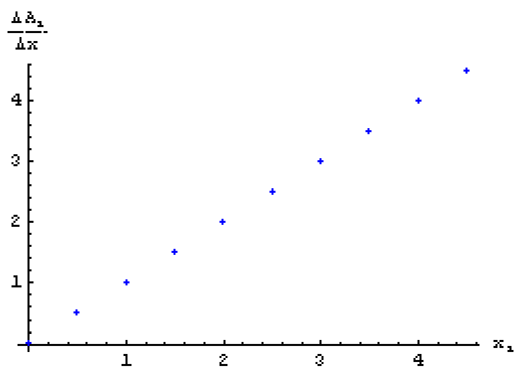
In[37]:=

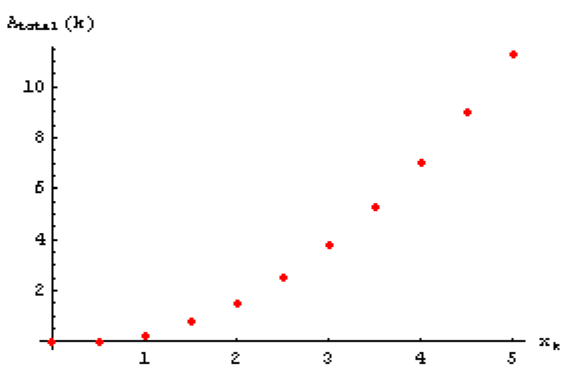
```
Clear[f, ΔA, a, b, h, n];

f[x_] = x;
```

```
a = 0;

b = 5;

n = 10;

fitlist = {1, x, x^2};

areas;
```

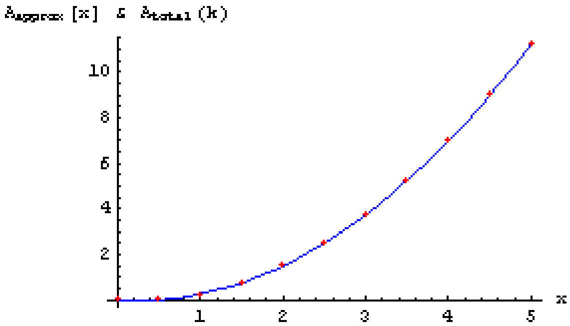
The ΔA[i] function is $\frac{i}{4}$

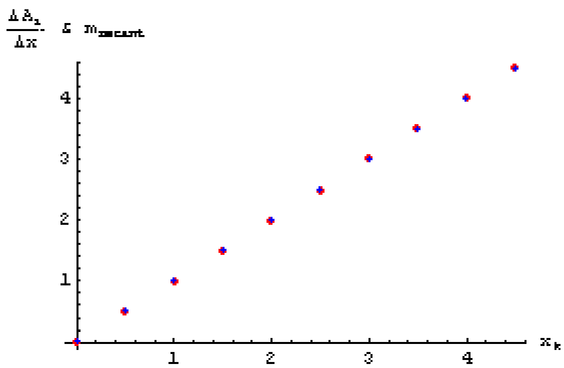
The $\frac{\Delta A[i]}{\Delta x}$ function is $\frac{i}{2}$





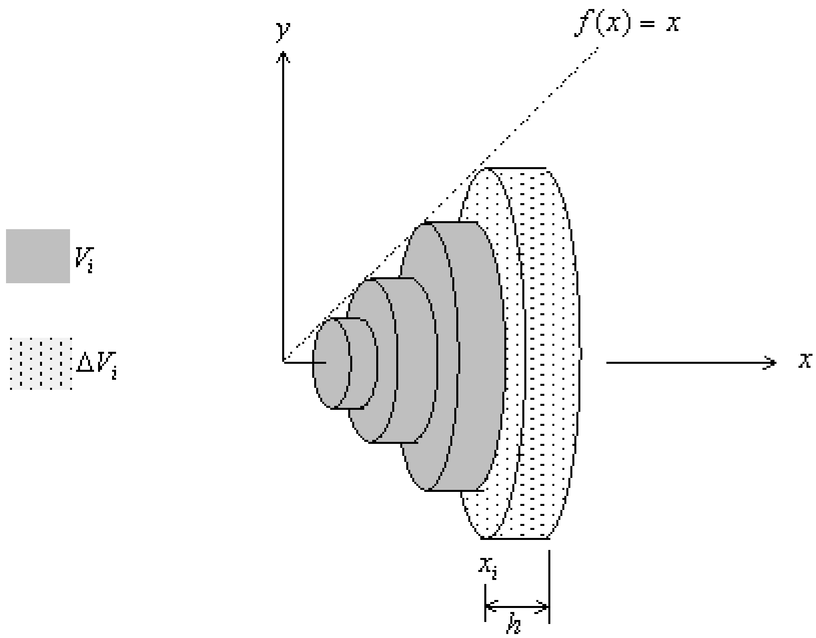
The function Aapprox[x] is
$-2.65704\times 10^{-16} - 0.25\,x + 0.5\,x^2$





Part II: Volumes

Chapter 6, Sections 1 and 2



Note: In this exercise, ΔV_i represents the volume of the i^{th} disk, as shown in the figure above.

Our first objective is to form a function, **volumeSum[k]**, that estimates the volume of the solid that is generated when the region bounded by the graph of a function f(x), the *x*-axis, and the vertical lines x = a and x = x_k =a+k h is rotated about the *x*-axis. We do this by adding the ΔV_i 's, the volumes of the first *k* disks, where *k* can range from 0 up to *n*, the number of disks of thickness *h* between x=a and x=b.

For the function *f*(*x*) = *x*, we write ΔV_i , the volume of the i^{th} disk, in terms of f(x_i) and *h*, where x_i = a+i h. At first, we let a =0, b=5, and n=10.

In[44]:=

```
Clear[f, ΔV, a, b, h, n];

f[x_] = x;

a = 0;

b = 5;

n = 10;

h = (b - a) / n;

ΔV[i_] = π (f[a + i * h])^2 * h
```

Out[50]=

$$\frac{i^2 \pi}{8}$$

The function $\frac{\Delta V(i)}{\Delta x}$ is the rate at which the total volume of the solid accumulates with each increment of *h* added to *x*. We generate a list of the rate-of-change values for each x_i =a+i h, plot the rate of change versus x_i , and then assign the graph to the symbol name **p0** for later reference.

In[51]:=

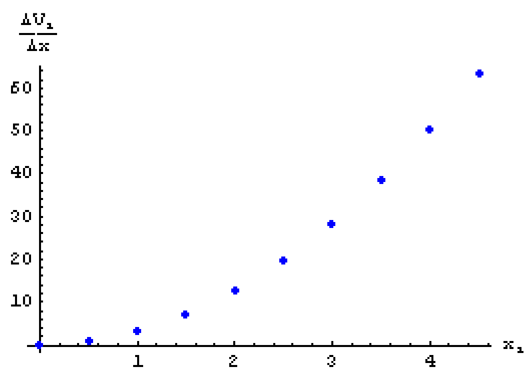
```
rateList = Table[{a + i h, ΔV[i]/h}, {i, 0, n - 1}]
```

Out[51]=

$$\left\{\{0,0\},\left\{\frac{1}{2},\frac{\pi}{4}\right\},\{1,\pi\},\left\{\frac{3}{2},\frac{9\pi}{4}\right\},\{2,4\pi\},\left\{\frac{5}{2},\frac{25\pi}{4}\right\},\{3,9\pi\},\left\{\frac{7}{2},\frac{49\pi}{4}\right\},\{4,16\pi\},\left\{\frac{9}{2},\frac{81\pi}{4}\right\}\right\}$$

In[52]:=

```
p0 = ListPlot[rateList,
  PlotStyle → {RGBColor[0, 0, 1], PointSize[0.
  AxesLabel → {"x_i", "ΔV_i/Δx"}];
```



You have probably noticed that, for the volume problem, the rate at which the total volume of the disks accumulates with each added increment *h* is $\pi\ f(x_i)^2$, the area of the circular end of the i^{th} disk. That is, $\frac{\Delta V(i)}{\Delta x} = \pi\ f(x_i)^2$.

Now we form a function that calculates the left Riemann sum of the ΔV_i's for the first *k* disks where *k* can range between 0 and n.

In[53]:=

$$\texttt{volumeSum[k\_] :=} \sum_{i=0}^{k-1} \texttt{ΔV[i];}$$

We form a list of the ordered pairs, in which the first element of each ordered pair is the *x*-coordinate of the left side of the last disk in the sum, and the second element in each ordered pair is the volume of the first *k* disks.

In[54]:=

```
volumelist = Table[{a + k * h, volumeSum[k]}, {k
```

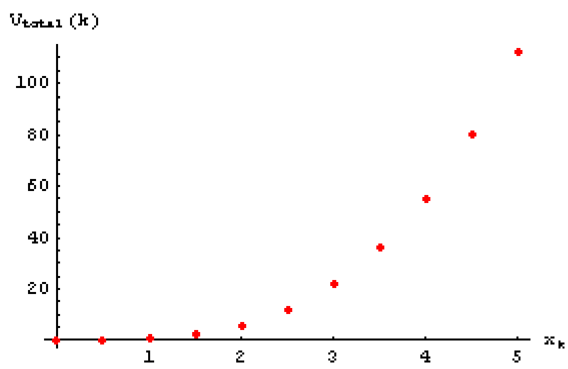
Out[54]=

$$\left\{\{0, 0\}, \left\{\frac{1}{2}, 0\right\}, \left\{1, \frac{\pi}{8}\right\}, \left\{\frac{3}{2}, \frac{5\pi}{8}\right\}, \left\{2, \frac{7\pi}{4}\right\}, \left\{\frac{5}{2}, \frac{15\pi}{4}\right\}, \left\{3, \frac{55\pi}{8}\right\}, \left\{\frac{7}{2}, \frac{91\pi}{8}\right\}, \left\{4, \frac{35\pi}{2}\right\}, \left\{\frac{9}{2}, \frac{51\pi}{2}\right\}, \left\{5, \frac{285\pi}{8}\right\}\right\}$$

Let's plot the list of points and assign it to the symbol name **p1** for later use.

In[55]:=

```
p1 = ListPlot[volumelist,
  PlotStyle → {RGBColor[1, 0, 0], PointSize[0.
  AxesLabel → {"xk", "Vtotal(k)"}, PlotRange → :
```



The function **volumeSum[k]** appears to be a cubic polynomial, so now we use the **Fit[ ]** command to see if there is a cubic function that passes through the points.

In[56]:=

$$V_{approx}[x\_] = \text{Fit}\left[\text{volumelist}, \{1, x, x^2, x^3\}, x\right]$$

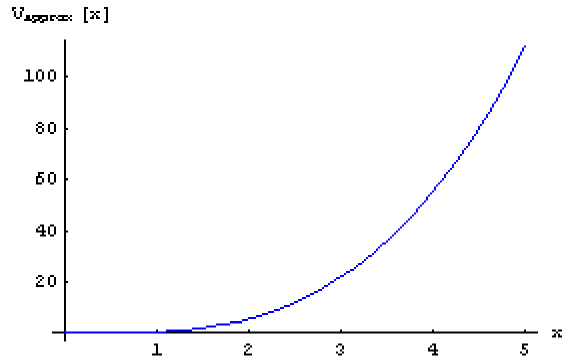
Out[56]=

$$-2.98598 \times 10^{-14} + 0.1309\,x - 0.785398\,x^2 + 1.0472\,x^3$$

Now we plot V_{approx}[x] and the points in **volumelist** to show that the points do in fact lie on the graph of the cubic polynomial. First, we plot V_{approx}[x] and then show the two graphs on the same plot.

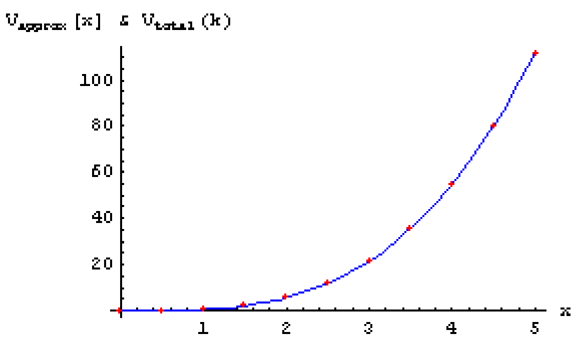
In[57]:=

```
p2 = Plot[Vapprox[x], {x, a, a + n h}, PlotRange -
  PlotStyle → {RGBColor[0, 0, 1]},
  AxesLabel → {"x", "Vapprox[x]"}];
```



In[58]:=

```
Show[{p2, p1}, AxesLabel → {"x", "Vapprox[x] &
```



The points appear to lie on the curve.

The slopes of the secant lines taken from the **volumeSum[k_]** graph give the rate at which the total area of the rectangles accumulates with each *h* increment of *x*. Now let's calculate the slopes of the secant lines between consecutive pairs of points on the graph above and plot them.

In[59]:=

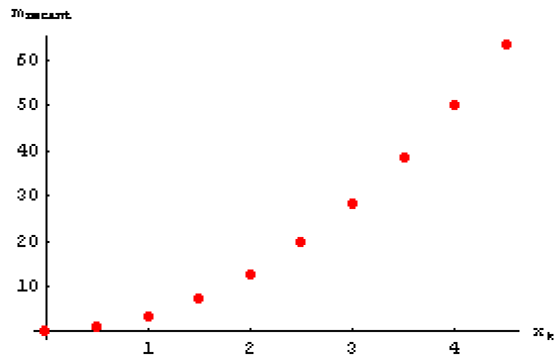
$$msecants = Table\left[\left\{a + k\,h, \frac{volumeSum[k+1] - vol}{h}\right.\right.$$
$$\left.\{k, 0, n-1\}\right]$$

Out[59]=

$$\left\{\{0, 0\}, \left\{\frac{1}{2}, \frac{\pi}{4}\right\}, \{1, \pi\}, \left\{\frac{3}{2}, \frac{9\pi}{4}\right\}, \{2, 4\pi\}, \left\{\frac{5}{2}, \frac{25\pi}{4}\right\}, \{3, 9\pi\}, \left\{\frac{7}{2}, \frac{49\pi}{4}\right\}, \{4, 16\pi\}, \left\{\frac{9}{2}, \frac{81\pi}{4}\right\}\right\}$$

In[60]:=

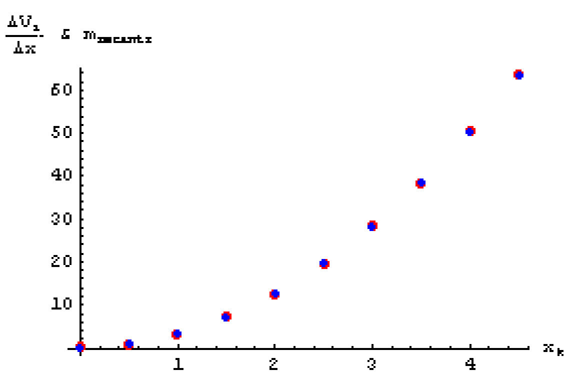
```
p4 = ListPlot[msecants,
  PlotStyle → {RGBColor[1, 0, 0], PointSize[0.
  AxesLabel → {"xk", "msecant"}];
```



But this is the same thing as $\frac{\Delta V(i)}{\Delta x}$ that we calculated at the beginning of this exercise. To confirm this, we show **msecants** and $\frac{\Delta V(i)}{\Delta x}$ together on the same graph.

In[61]:=

```
Show[{p4, p0}, AxesLabel → {"xk", "ΔVi/Δx & msec
```



It looks as though we've come full circle again.

You Try It: Part II - Taking it to the Limit and Another Function

Chapter 6, Sections 1 and 2

To help you with these exercises, we again copy all of the commands from Part II into a single cell, the one that follows. In each exercise, you are asked to change some of the red entries and answer the questions. (Note that we have hidden all but the input commands and put them in a command called **volumes**, the last command in the following cell. In addition, we suppress displaying the lists of values that are generated because they become too large when *n* is a big number.)

1. For the function f(x)=x, considered in Part II, increase *n*, the number of disks between x=a and x=b, so that the total volume of the disks approaches the volume of revolution. Try n=25, 50, 100, 250, 500. (For larger values of *n*, the evaluation of the commands will take a while because of the large number of ΔV_i 's that must be added to form each value of **volumeSum [k]**.) As *n* gets larger, what function does **V[x]** appear to be approaching? Also, as *n* gets larger, Δ x=h gets closer to 0. What is the $\lim_{\Delta x \to 0} \frac{\Delta V_i}{\Delta x}$? In the limit as n→∞ and h→0, what is the relationship between $\frac{\Delta V_i}{\Delta x}$ and **V[x]**?

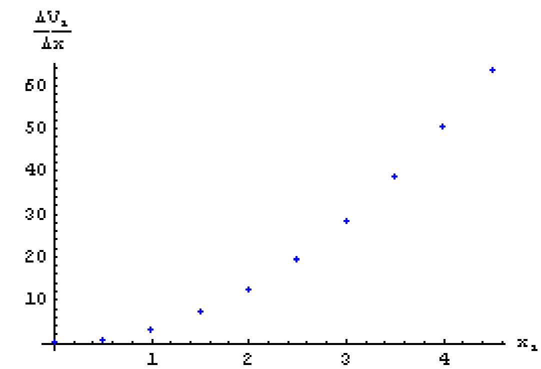
2. Repeat Exercise 1 for the following: f(x)=sin x, a=0, b=2π, and **fitlist** ={1,x,sin 2x,cos 2x}. Explain the stair-step pattern in the graph of **V[x]**, and explain why we chose the functions in **fitlist** for the **Fit[ ]** command.

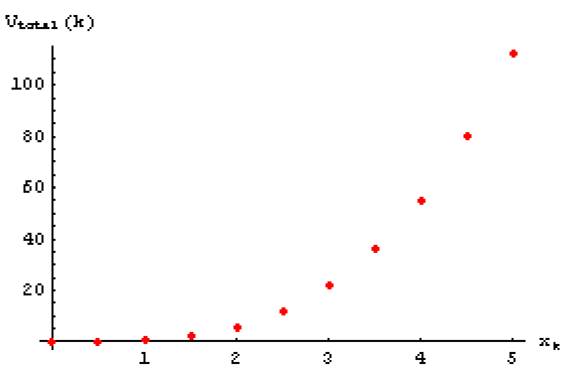
In[62]:=

```
Clear[f, ΔV, a, b, h, n];

f[x_] = x;

a = 0;

b = 5;

n = 10;

fitlist = {1, x, x^2, x^3};

volumes;
```

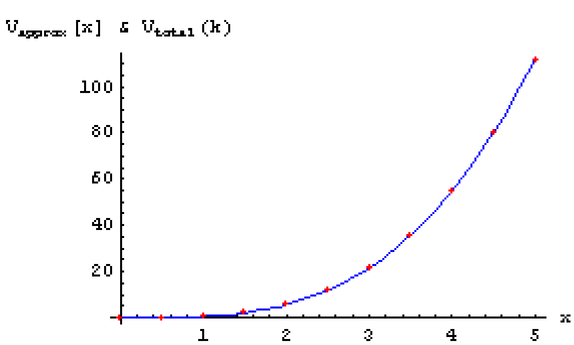
The ΔV[i] function is $\frac{i^2 \pi}{8}$

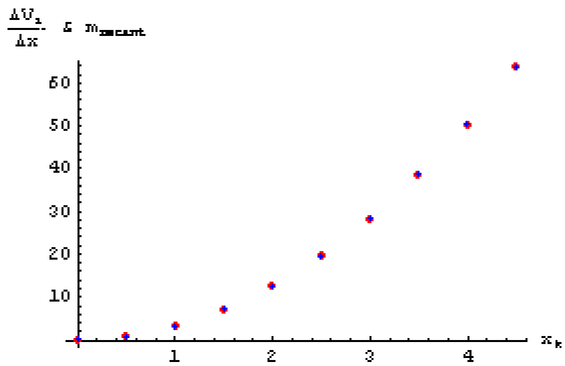
The $\frac{\Delta V[i]}{\Delta x}$ function is $\frac{i^2 \pi}{4}$





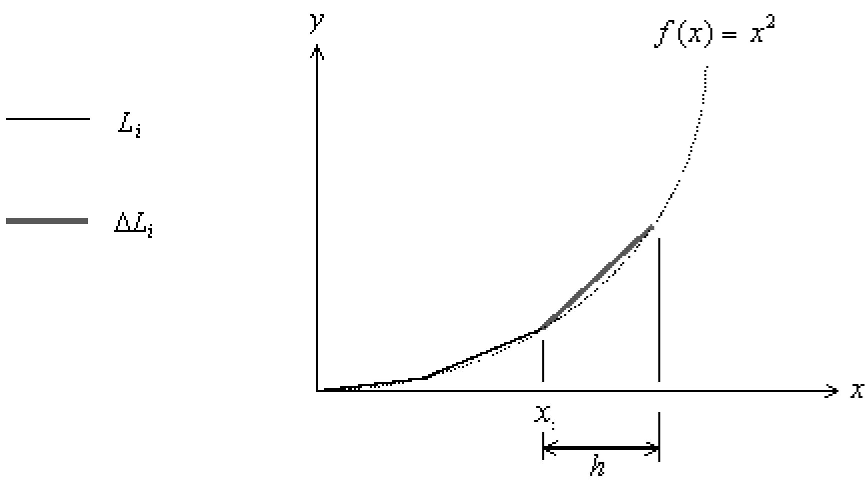
The function Vapprox[x] is $-2.98096 \times 10^{-14} + 0.1309\, x - 0.785398\, x^2 + 1.0472\, x^3$





Part III: Lengths of Arc

Chapter 6, Section 3



Note: In this exercise, ΔL_i represents the length of the i^{th} line segment, as shown in the figure above.

Our first objective is to form a function, **lengthSum[k]**, that estimates the length of the graph of a function, f(x) as *x* varies from x = a and x = x_k =a+k h. We do this by adding the ΔL_i 's, the lengths of the first *k* line segments, where *k* can range from 0 up to *n*, the number of line segments between x=a and x=b.

For the function $f(x) = x^2$, we need to write ΔL_i in terms of $f(x_i)$, $f(x_i + h)$, and *h*, where $x_i = a + ih$. To do this, we use the Pythagorean theorem to calculate the length of the line segment that begins at the point (x_i , $f(x_i)$) and ends at the point (x_i +h, $f(x_i + h)$).

$$\Delta L_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{h^2 + \left(\frac{\Delta y_i}{h}\right)^2 h^2} = h\sqrt{1 + \left(\frac{\Delta y_i}{h}\right)^2} = h\sqrt{1 + \left(\frac{f(x_i + h) - f(x_i)}{h}\right)^2}.$$

Now let's put it into *Mathematica* code. We let a =-5, b=5, and n=10.

In[69]:=

```
Clear[f, ΔL, a, b, h, n];

f[x_] = x^2;

a = -5;

b = 5;

n = 10;

h = (b - a) / n;

ΔL[i_] = h Sqrt[1 + ((f[a + i h + h] - f[a + i h])/h)^2]
```

Out[75]=

$$\sqrt{1 + (-(-5 + i)^2 + (-4 + i)^2)^2}$$

The function $\frac{\Delta L(i)}{\Delta x}$ is the rate at which the total length of the line segments accumulates with each increment of *h* added to *x*. We generate a list of the rate-of-change values for each x_i =a+i h, plot the rate of change versus x_i, and then assign the graph to the symbol name **p0** for later reference.

In[76]:=

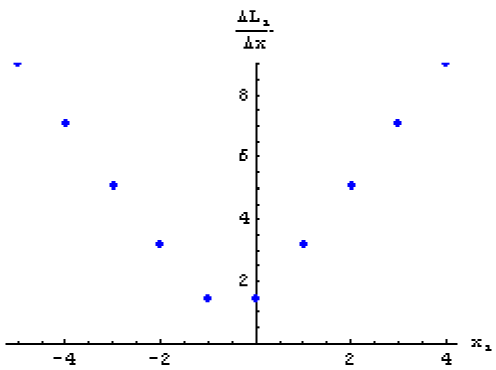
```
rateList = Table[{a + i h, ΔL[i]/h}, {i, 0, n - 1}]
```

Out[76]=

$$\{\{-5, \sqrt{82}\}, \{-4, 5\sqrt{2}\}, \{-3, \sqrt{26}\}, \{-2, \sqrt{10}\}, \{-1, \sqrt{2}\}, \{0, \sqrt{2}\}, \{1, \sqrt{10}\}, \{2, \sqrt{26}\}, \{3, 5\sqrt{2}\}, \{4, \sqrt{82}\}\}$$

In[77]:=

```
p0 = ListPlot[rateList, PlotRange → {0, Max[ra
  PlotStyle → {RGBColor[0, 0, 1], PointSize[0.
  AxesLabel → {"xᵢ", "ΔLᵢ/Δx"}];
```



The rate at which the total length of the line segments accumulates with each added increment *h* is

$$\sqrt{1+\left(\frac{f(x_i+h)-f(x_i)}{h}\right)^2}\text{, that is, }\frac{\Delta L(i)}{\Delta x}=\sqrt{1+\left(\frac{f(x_i+h)-f(x_i)}{h}\right)^2}.$$

Now we form a function that calculates the left Riemann sum of the ΔL_i's for the first *k* line segments where *k* can range between 0 and n.

In[78]:=

$$\texttt{lengthSum[k\_] :=} \sum_{i=0}^{k-1} \texttt{ΔL[i];}$$

We form a list of the ordered pairs, in which the first element of each ordered pair is the *x*-coordinate of the left side of the last line segment in the sum, and the second element in each ordered pair is the length of the first *k* line segments.

In[79]:=

```
lengthlist = Table[{a + k * h, lengthSum[k]}, {k
```

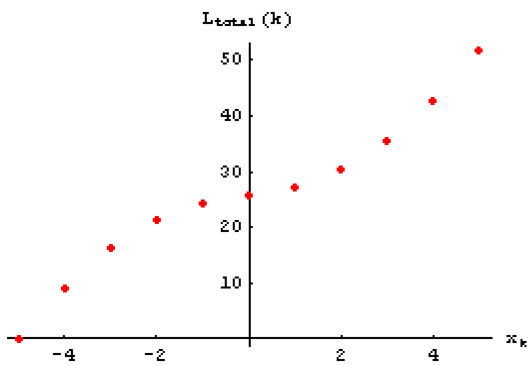
Out[79]=

$$\{\{-5, 0\}, \{-4, \sqrt{82}\}, \{-3, 5\sqrt{2}+\sqrt{82}\},$$
$$\{-2, 5\sqrt{2}+\sqrt{26}+\sqrt{82}\},$$
$$\{-1, 5\sqrt{2}+\sqrt{10}+\sqrt{26}+\sqrt{82}\},$$
$$\{0, 6\sqrt{2}+\sqrt{10}+\sqrt{26}+\sqrt{82}\},$$
$$\{1, 7\sqrt{2}+\sqrt{10}+\sqrt{26}+\sqrt{82}\},$$
$$\{2, 7\sqrt{2}+2\sqrt{10}+\sqrt{26}+\sqrt{82}\},$$
$$\{3, 7\sqrt{2}+2\sqrt{10}+2\sqrt{26}+\sqrt{82}\},$$
$$\{4, 12\sqrt{2}+2\sqrt{10}+2\sqrt{26}+\sqrt{82}\},$$
$$\{5, 12\sqrt{2}+2\sqrt{10}+2\sqrt{26}+2\sqrt{82}\}\}$$

Let's plot the list of points and assign it to the symbol name **p1** for later use.

In[80]:=

```
p1 = ListPlot[lengthlist,
  PlotStyle → {RGBColor[1, 0, 0], PointSize[0.
  AxesLabel → {"xk", "Ltotal(k)"}, PlotRange → :
```



For comparison, we calculate the exact arc-length function, $L_{exact}(x)$, by integrating $\frac{dL}{du} = \sqrt{1+f'(u)^2}$ from u=a, to u=x.

In[81]:=

```
Lexact[x_] = Assuming[x ∈ Reals, ∫_a^x √(1 + f′[u]²) (
```

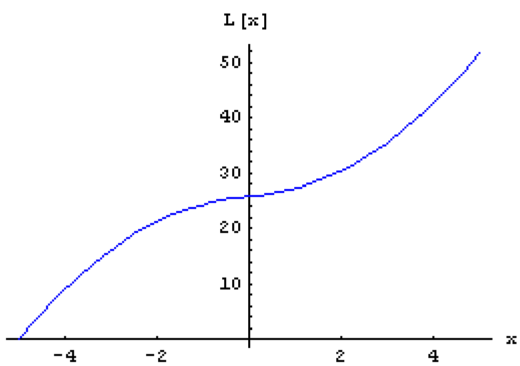
Out[81]=

```
If[x > -5, 1/4 (10 √101 + 2 x √(1 + 4 x²) +
    ArcSinh[10] + ArcSinh[2 x]),
  Integrate[√(1 + 4 u²), {u, -5, x},
   Assumptions → x ≤ -5]]
```

Now let's plot the function and then show the **lengthSum[k]** points and the function on the same graph.

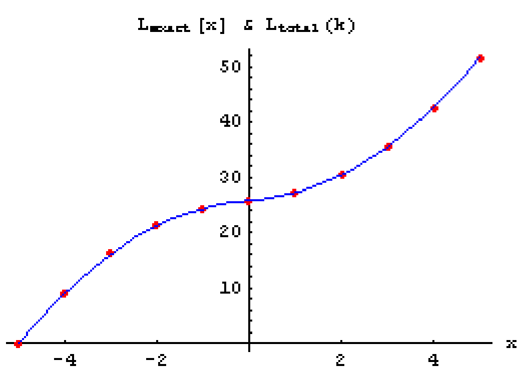
In[82]:=

```
p2 = Plot[L_exact[x], {x, a, a + n * h}, PlotStyle -
   PlotRange → All, AxesLabel → {"x", "L[x]"}];
```



In[83]:=

```
Show[p1, p2, AxesLabel → {"x", "L_exact[x] & L_tc
```



The slopes of the secant lines taken from the **lengthSum[k_]** graph give the rate at which the total length of the straight-line segments accumulates with each h increment of x. Now let's calculate the slopes of the secant lines between consecutive pairs of points on the graph above and plot them.

In[84]:=

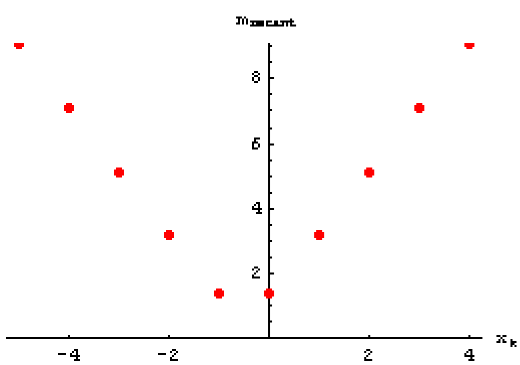
```
msecants = Table[{a + k h, (lengthSum[k + 1] - le
                              h
  {k, 0, n - 1}]
```

Out[84]=

$$\{\{-5, \sqrt{82}\}, \{-4, 5\sqrt{2}\}, \{-3, \sqrt{26}\}, \{-2, \sqrt{10}\}, \{-1, \sqrt{2}\}, \{0, \sqrt{2}\}, \{1, \sqrt{10}\}, \{2, \sqrt{26}\}, \{3, 5\sqrt{2}\}, \{4, \sqrt{82}\}\}$$

In[85]:=

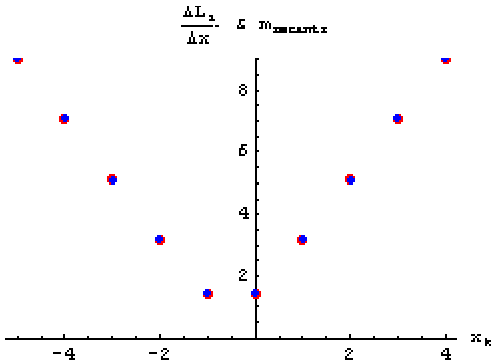
```
p4 = ListPlot[msecants, PlotRange → {0, Max[ms
  PlotStyle → {RGBColor[1, 0, 0], PointSize[0.
  AxesLabel → {"xk", "msecant"}];
```



But this is the same thing as $\frac{\Delta L(i)}{\Delta x}$ that we calculated at the beginning of this exercise. To confirm this, we show **msecants** and $\frac{\Delta L(i)}{\Delta x}$ together on the same graph.

In[86]:=

```
Show[{p4, p0}, AxesLabel → {"xk", "ΔLi/Δx & msec
```



Yet again, it looks as though we've come full circle.

You Try It: Part III - Taking it to the Limit and Another Function

Chapter 6, Section 3

To help you with these exercises, we again copy all of the commands from Part III into a single cell, the one that follows. In each exercise, you are asked to change some of the red entries and answer the questions. (Note that we have hidden all but the input commands and put them in a command called **lengths**, the last command in the following cell. In addition, we suppress displaying the lists of values that are generated because they become too large when *n* is a big number.)

1. For the function f(x)=x^2, considered in Part III, increase *n*, the number of straight-line segments between x=a and x=b, so that the total length of the line segments approaches the length of the graph of f(x) for a≤x≤b. Try n=25, 50, 100, 250, 500. (For larger values of *n*, the evaluation of the commands will take a while because of the large number of ΔL_i's that must be added to form each value of **lengthSum[k]**.) What is the $\lim_{\Delta x \to 0} \frac{\Delta L_i}{\Delta x}$? In the limit as n→∞ and h→0, what is the relationship between $\frac{\Delta L_i}{\Delta x}$ and **L_{exact}[x]**?

2. Repeat Exercise 1 for the following: f(x)=5sin (2x), a=0, b=π. In addition, explain the stair-step pattern in the graph of **L_{exact}[x]**. Under what conditions does the accumulated length of the straight-line segments increase the slowest with each increment *h* added to *x*, and under what conditions does it increase most rapidly?

In[87]:=

```
Clear[f, ΔL, a, b, h, n];
```

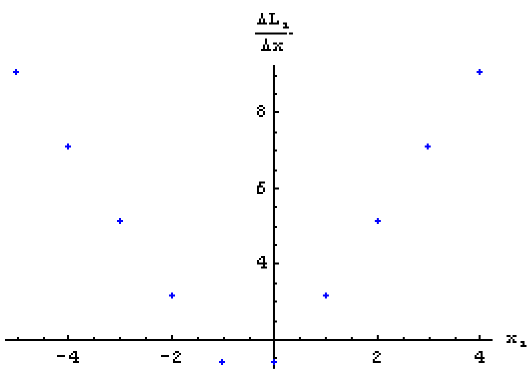
```
f[x_] = x^2;

a = -5;

b = 5;

n = 10;

lengths;
```

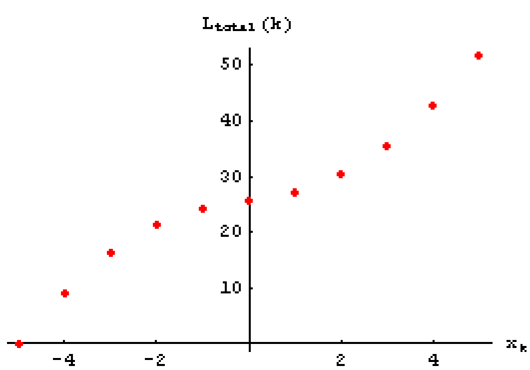
The ΔL[i] function is

$$\sqrt{1+(-(-5+i)^2+(-4+i)^2)^2}$$

The $\frac{\Delta L[i]}{\Delta x}$ function is

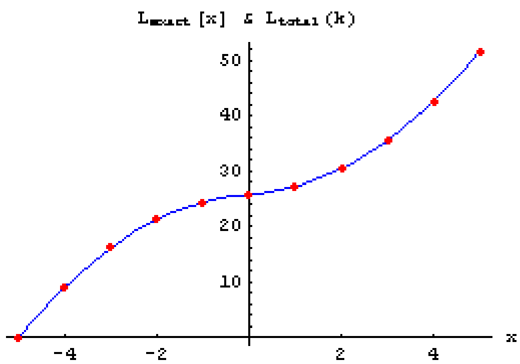
$$\sqrt{1+(-(-5+i)^2+(-4+i)^2)^2}$$

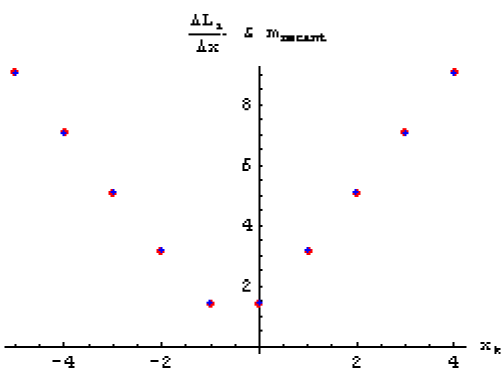




The function Lexact[x] is

$$\text{If}\left[x > -5,\ \frac{1}{4}\left(10\sqrt{101} + 2x\sqrt{1+4x^2} + \text{ArcSinh}[10] + \text{ArcSinh}[2x]\right),\ \text{Integrate}\left[\sqrt{1+4u^2},\ \{u, -5, x\},\ \text{Assumptions} \to x \le -5\right]\right]$$





You probably noticed that the arc length integrals for the function f(x)=5sin(2x) are not elementary, giving **EllipticE[ ]** functions. Unlike most of the area and volume integrals that we consider in beginning calculus, the arc length integrals for elementary functions can easily give rise to non-elementary integrals.

Created by *Mathematica* (August 9, 2005)