Visualizing and Interpreting the Divergence Theorem

Note: You may notice differences between this Maple worksheet and the equivalent Mathematica notebook. These differences were introduced to preserve the content of these modules and were necessary because of major functional differences between Maple and Mathematica.

Introduction

OBJECTIVE: Observe that surface integrals are difficult to evaluate, even using *Maple*. See that using parametrizations to evaluate flux surface integrals and applying the Divergence Theorem can help with integral evaluations.

You will see that surface integrals are still difficult to set up and to evaluate, even with *Maple*, but parametrizations can assist in evaluating these difficult flux surface integrals. You will also verify the Divergence Theorem, just as you have done with Green's Theorem. This module is an example. However, those familiar with *Maple* can adjust the code to solve other problems.

Technology Guidelines

NOTE: If you have just finished a worksheet, **restart** *Maple* before executing a new worksheet.
TO OPEN SECTIONS,
Click on the **PLUS** sign at the left hand side of the screen *or* select **Expand All Sections** from the **View** drop down menu.

TO STOP AN EXECUTION
Click on **STOP** button from the toolbar.

ORDER OF EXECUTION
Execute commands in the order given. Do not skip any *Maple* Input lines within a given worksheet

Alternatively, you can execute the entire worksheet by selecting the **Execute Worksheet** command from the **Edit** drop down menu.

SAVING WORKSHEETS.

You can save anytime to any directory you choose, and it is wise to save often.
EXPERIENCING MAJOR PROBLEMS
Save if appropriate, and then shut down *Maple* and start it up.

Part I: Visualizing the Problem

We select the object formed by a hemisphere of radius 1, topped with a cylinder of radius 1 and height 1, with a circular top on top on the cylinder. The object is similar to the one to which the his is the method of moments was applied in the previous chapter. The force we choose has components $[-x^2 - 4\,x\,y, -6\,y\,z, 12\,z]$.

```
> restart;
```

```
> force:=[-x^2-4*x*y, -6*y*z, 12*z];
```

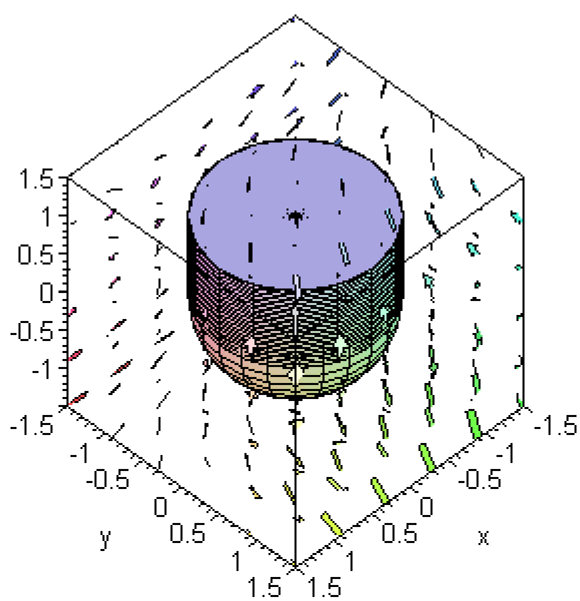
$$force := [-x^2 - 4\,x\,y, -6\,y\,z, 12\,z]$$

Before we apply the Divergence Theorem, we plot the figure. We need to load in a graphing packages to show our plots.

```
> with(plots):
```

Warning, the name changecoords has been redefined

```
> forceplot:=fieldplot3d(force, x=-1.5..1.5, y=-1.5..1.5, z=-1.5...1.5, axes=BOXED, grid=[6,6,6
  arrows=THICK):
  hemisplot:=sphereplot(1, r=0..2*Pi, theta=0..Pi, labels=["x","y","z"], view = [-1..1,-1..1,-1.
  axes=boxed):
  pcyl:=plottools[cylinder]([0,0,1],1,.01):
  #cylinder:=cylinderplot(1, r=0..2*Pi, theta=0..2*Pi, view=[-1..1,-1..1,-1..1]):
  topplot := plot3d([cos(t),sin(t),u],t=-2*Pi..Pi, u=0..1):
  display({forceplot, hemisplot, pcyl, topplot}, view=[-1.5..1.5, -1.5..1.5, -1.5..1.5]);
```



Part II: Evaluating the Divergence Integral

The divergence integral over the volume is easy to set up and calculate. First, we begin by loading in a package that performs dot products and cross products.

```
> with(linalg):
```

Warning, the protected names norm and trace have been redefined and unprotected

```
> unassign('x','y','z');
  divf:=diff(force[1], x)+diff(force[2],y)+diff(force[3],z):
  print(`The divergence of this force is`, divf);
  int(int(int(divf, z=-sqrt(1-x^2-y^2)..1), y=-sqrt(1-x^2)..sqrt(1-x^2)), x=-1..1):
  totalflux:=evalf(%):
  print(`The value of the divergence integral is`,totalflux);
```

$$\textit{The divergence of this force is}, -2\,x - 4\,y - 6\,z + 12$$

$$\textit{The value of the divergence integral is}, 58.11946410$$

Part III: Finding Surface Integrals

The surface integrals evaluating the flux across each surface require considerable set up. As you recall, parametrizations are very convenient in determining line integrals. Likewise, parametrizations can facilitate the computation of surface integrals. We use spherical coordinates to write the equation of the sphere parametrically and follow the procedure outlined in **Section 12.6** of your text.

Hemisphere

```
> unassign('x','y','z','r','theta','phi');
  x:=sin(theta)*sin(phi):
  y:=cos(theta)*sin(phi):
  z:=cos(phi):
  r:=[x,y,z]:
  rtheta:=diff(r,theta):
  rphi:=diff(r,phi):
  cr:=linalg[crossprod](rtheta, rphi):
  integrand:=12*cos(phi)^2*sin(phi)-6*cos(theta)^2*cos(phi)*sin(phi)^3-sin(theta)^2*
  (theta)+sin(theta))*sin(phi)^4:
  print(`The integrand for the hemispherical surface integral is`, integrand);
  hemisflux:=int(int(integrand, phi=(Pi/2)..Pi), theta=0..2*Pi):
  print(`The value of the hemispherical surface integral is`, hemisflux,`that is`,evalf(he
```

The integrand for the hemispherical surface integral is,

$$12\cos(\phi)^2\sin(\phi) - 6\cos(\theta)^2\cos(\phi)\sin(\phi)^3 - \sin(\theta)^2\,(4\cos(\theta)+\sin(\theta))\sin(\phi)^4$$

$$\textit{The value of the hemispherical surface integral is}, \frac{19\,\pi}{2}, \textit{that is}, 29.84513021$$

The equations of the cylinder and the top can be written using polar coordinates, with z and θ the parameters for the cylinder.

Cylinder

```
> unassign('x','y','z','r');
  x:=cos(theta):
  y:=sin(theta):
  z:=z:
  r:=[x,y,z]:
  rtheta:=diff(r,theta):
  rz:=diff(r,z):
  cr:=linalg[crossprod](rtheta, rz):
  integrand:=-cos(theta)^3-4*cos(theta)^2*sin(theta)-6*z*sin(theta)^2:
  print(`The integrand for the cylindrical surface integral is`, integrand);
  cylflux:=int(int(integrand, z=0..1), theta=0..2*Pi):
  print(`The value of the cylindrical surface integral is`, cylflux, `that is`, evalf(cylflux)
```

The integrand for the cylindrical surface integral is, $-\cos(\theta)^3 - 4\cos(\theta)^2 \sin(\theta) - 6 z \sin(\theta$

The value of the cylindrical surface integral is, -3π, *that is,* -9.424777962

Top

Polar coordinates r and θ will be the parameters for the top. Recall that for the top, the normal is always in the increasing z direction. Also the delta area form of the area must contain the polar coordinate r, which varies from 0 to 1.

```
> unassign('x','y','z','r');
  x:=cos(theta):
  y:=sin(theta):
  z:=1:
  r:=[x,y,z]:
  norm1:=[0,0,1]:
  integrand:=linalg[dotprod](force,norm1):
  print(`The integrand for the top surface integral is`, integrand);
  topflux:=int(int(R*integrand, R=0..1),theta=0..2*Pi):
  print(`The value of the surface integral for the top is`, topflux, `that is` ,evalf(topflux
```

The integrand for the top surface integral is, 12

The value of the surface integral for the top is, 12π, *that is,* 37.69911185

Part IV: Verifying the Divergence Theorem

Compare your answers to verify the Divergence Theorem.

```
> totalsurfaceflux := hemisflux + cylflux + topflux;
  print(`The total of the surface integrals is `, totalsurfaceflux, `that is`,evalf(totalsurfaceflux
  if (evalf(totalsurfaceflux)<=(totalflux+0.001) and evalf(totalsurfaceflux)>=totalflux-.001) tl
  print(`The Divergence Theorem is validated.`) else print(`there's a problem`) fi;
```

$$totalsurfaceflux := \frac{37\,\pi}{2}$$

$$\text{The total of the surface integrals is },\ \frac{37\,\pi}{2},\ \text{that is},\ 58.11946410$$

The Divergence Theorem is validated.

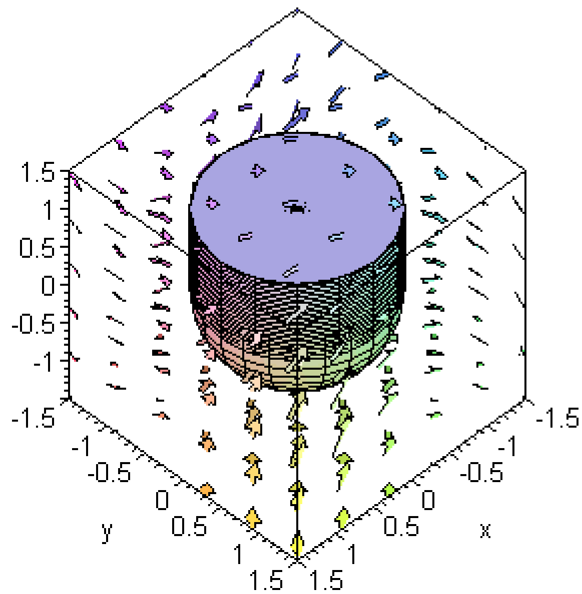
You Try It: Part IV

Visualize and Evaluate Other Divergence Integrals

You can change the force function to define a different force function and then to visualize it.

```
> unassign('x','y','z', 't', 'r');
  force:=[cos(10*z), sin(10*z), x*y];
  forceplot:=fieldplot3d(force, x=-1.5..1.5, y=-1.5..1.5, z=-1.5...1.5, axes=BOXED, grid=
  arrows=THICK):
  hemisplot:=sphereplot(1, r=0..2*Pi, theta=0..Pi, labels=["x","y","z"], view = [-1..1,-1
  axes=boxed):
  pcyl:=plottools[cylinder]([0,0,1],1,.01):
  #cylinder:=cylinderplot(1, r=0..2*Pi, theta=0..2*Pi, view=[-1..1,-1..1,-1..1]):
  topplot := plot3d([cos(t),sin(t),u],t=-2*Pi..Pi, u=0..1):
  display({forceplot, hemisplot, pcyl, topplot}, view=[-1.5..1.5, -1.5..1.5, -1.5..1.5]);
```

$$force := [\cos(10\,z),\ \sin(10\,z),\ x\,y]$$




```
> unassign('x','y','z');
  divf:=diff(force[1], x)+diff(force[2],y)+diff(force[3],z):
  print(`The divergence of this force is`, divf);
  totalflux:=int(int(int(divf, z=-sqrt(1-x^2-y^2)..1), y=-sqrt(1-x^2)..sqrt(1-x^2)), x=-1..1
  print(`The value of the divergence integral is`, totalflux);
```

The divergence of this force is, 0

The value of the divergence integral is, 0

Selection of Closed Region to Maximize Flux

Section 16.8, Exercise 24

In this problem, you are to find the dimensions of a particular rectangular box for which the total flux of a force with components $[-x^2 - 4\,x\,y, -6\,y\,z, 12\,z]$ is maximized.

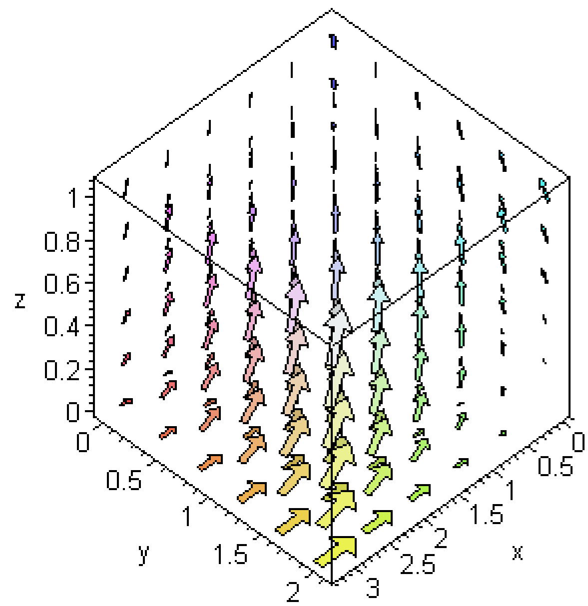
Visualizing the Problem

The force here is the same as in the previous part, but we are now asked to find the coordinates of a rectangular box that will maximize the outward flux. Before we apply the Divergence Theorem, let's plot the figure. Before we apply the Divergence Theorem, let's plot the figure. Remember, you can view the plot from different angles by clicking on the plot and draggin the mouse.

```
> unassign('x','y','z','t','force');
```

```
force:=[-x^2-4*x*y, -6*y*z, 12*z];
forceplot:=fieldplot3d(force, x=0..3, y=0..2, z=0..1, labels=["x","y","z"], axes=
[ 6,6,6], arrows=THICK):
display(forceplot);
```

$$force := [-x^2 - 4\,x\,y, -6\,y\,z, 12\,z]$$



Evaluating and Maximizing the Divergence Integral

```
> divf:=diff(force[1], x)+diff(force[2],y)+diff(force[3],z):
  divergenceintegral:=int(int(int(divf, z=0..1), y=0..b), x=0..a);
```

$$divergenceintegral := -b\,a^2 - 2\,b^2\,a + 9\,b\,a$$

We will find the values of *a* and *b* that maximize this function by computing the two first partials with respect to *a* and *b* and equating them both to 0.

```
> ma:=diff(divergenceintegral,a):
  mb:=diff(divergenceintegral,b):
  solab:=solve({ma=0, mb=0},{a,b});
```

$$solab := \{b = 0, a = 0\}, \{a = 0, b = \frac{9}{2}\}, \{a = 9, b = 0\}, \{a = 3, b = \frac{3}{2}\}$$

Based on these results, what answer will you give? Evaluate the divergence integral at these values for *a* and *b*.

```
> solab[1];
  eval(divergenceintegral, solab[1]);
```

$$\{b = 0, a = 0\}$$

$$0$$

```
> solab[2];
  eval(divergenceintegral, solab[2]);
```

$$\{a = 0, b = \frac{9}{2}\}$$

$$0$$

```
> solab[3];
  eval(divergenceintegral, solab[3]);
```

$$\{a = 9, b = 0\}$$

$$0$$

```
> solab[4];
  eval(divergenceintegral, solab[4]);
```

$$\{a = 3, b = \frac{3}{2}\}$$

$$\frac{27}{2}$$

```
> ?

>
```

You should check this integral out for other values of a and b to convince yourself that you have maximized the divergence integral. Does the fact that this is less than the divergence integral found in Part II mean that you have not maximized the divergence integral as requested?