How Does Heat Dissipate?

Note: You may notice differences between this Maple worksheet and the equivalent Mathematica notebook. These differences were introduced to preserve the content of these modules and were necessary because of major functional differences between Maple and Mathematica.

Introduction

OBJECTIVE: Observe the physical interpretations of the contour and level curve plots and see an application of the Fourier Series.

Do you want to see physical interpretations of the contours and level curves you have been drawing? This equation of the heat will enable you to do so. The heat equation is a partial differential equation and its solution employs the Fourier series in a meaningful way.
Note: Part II requires knowledge of the Fourier series.

Technology Guidelines

NOTE: If you have just finished a worksheet, **restart** *Maple* before executing a new worksheet.
TO OPEN SECTIONS,
Click on the **PLUS** sign at the left hand side of the screen *or* select **Expand All Sections** from the **View** drop down menu.

TO STOP AN EXECUTION
Click on **STOP** button from the toolbar.

ORDER OF EXECUTION
Execute commands in the order given. Do not skip any *Maple* Input lines within a given worksheet

Alternatively, you can execute the entire worksheet by selecting the **Execute Worksheet** command from the **Edit** drop down menu.

SAVING WORKSHEETS.

You can save anytime to any directory you choose, and it is wise to save often.
EXPERIENCING MAJOR PROBLEMS
Save if appropriate, and then shut down *Maple* and start it up again.

Background

The following problems represent solutions to the partial differential heat equation:

$w_{xx} = \frac{1\ w_t}{c^2}$, where *w(x,t)* represents the temperature at position *x* at time *t* in a thin rod of length *L* with perfectly insulated sides. The value of the positive constant c^2 is determined by the material from which the rod is made.

Part 1: Fixed End Temperatures

We first examine a very simple solution to the problem where we assume that the ends of the rod are immersed in ice and are maintained at a temperature of zero degrees centigrade. If the rod is of length 4, the 0 degrees centigrade. at both ends is represented by the boundary conditions w(0,t) = w(4,t) = 0. Assume also that the initial temperature distribution within the rod is given by $w(x, 0) = \sin(2\pi x)$.

We first show that $w(x, t) = e^{(-3t)} \sin(2\pi x)$ is a solution to the differential equation with these boundary and initial conditions. Also determine the value of the constant *c* for this problem.

```
> restart;
w:=(x,t)->exp(-3*t)*sin(2*Pi*x):
print(cat(`The temperature at the left end of the rod is `, w(0,t)));
print(cat(`The temperature at the right end of the rod is `, w(4,t)));
print(cat(`The initial temperature distribution is `, w(x,0)));
print(cat(`The second derivative of the temperature with respect to position is `, diff(diff(w
x), x)));
wxx:= diff(diff(w(x,t), x), x):
print(cat(`The first derivative of the temperature with respect to time is `, diff(w(x,t), t)));
wt:=diff(w(x,t), t):
c:=sqrt(wt/wxx):
print(cat(`c =`, c, ` or `, evalf(c)));
if wxx = 1/c^2*wt then
print(cat(w(x,t), ` satisifies the differential equation.`))
else
print(`The equation is not satisfies the differential equation.`);
fi;
```

The temperature at the left end of the rod is 0

The temperature at the right end of the rod is 0

The initial temperature distribution is $\|(\sin(2\pi x))$

The second derivative of the temperature with respect to position is $\|(-4\,\mathbf{e}^{(-3t)} \sin(2\pi x)\,\pi^2)$

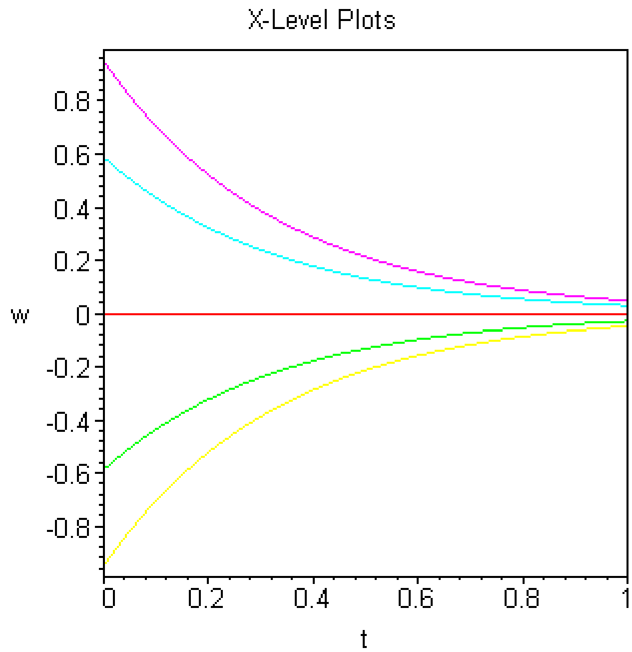
The first derivative of the temperature with respect to time is $\|(-3\,\mathbf{e}^{(-3\,t)}\sin(2\,\pi\,x))$

$$c = \|\left(\frac{\sqrt{3}}{2\,\pi}\right)\| \textit{ or } \|(0.2756644478)$$

$\|(\mathbf{e}^{(-3\,t)}\sin(2\,\pi\,x))\|$ *satisifies the differential equation.*

Plot the temperature function at various positions along the rod as a function of time.

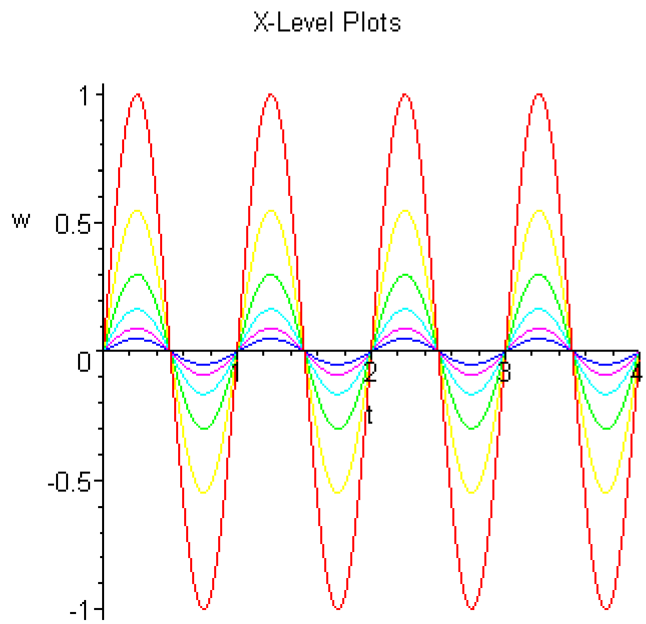
```
> plot([w(0, t), w(0.8, t), w(1.6, t), w(2.4, t), w(3.2, t), w(4, t)], t=0..1, color=[COLOR(RGB,1,0,0),COLOR(RGB,1,1,0),COLOR(RGB,0,1,0),COLOR(RGB,0,1,1),COLOR(RGB,1,0,1),COLOR(RGB,0,0,1)], labels=[t, w], title=`X-Level Plots`,axes=boxed);
```



Interpret these plots. What is happening to the temperature along different parts of the rod?

Plot the temperature function at various times as a function of the position on the rod.

```
> plot([w(x, 0), w(x, 0.2), w(x, 0.4), w(x, 0.6), w(x, 0.8), w(x, 1)], x=0..4, color=[COLOR(RGB,1,0,0),COLOR(RGB,1,1,0),COLOR(RGB,0,1,0),COLOR(RGB,0,1,1),COLOR(RGB,1,0,1),COLOR(RGB,0,0,1)], labels=[t, w], title=`X-Level Plots`);
```



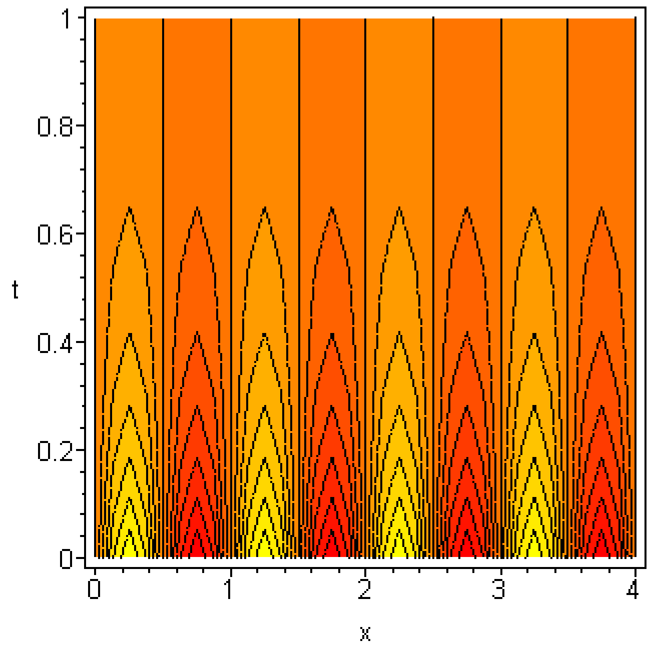
Interpret the results. What is happening to the temperature function as time goes on?

Plot the contours for the temperature function and the temperature function as a surface.

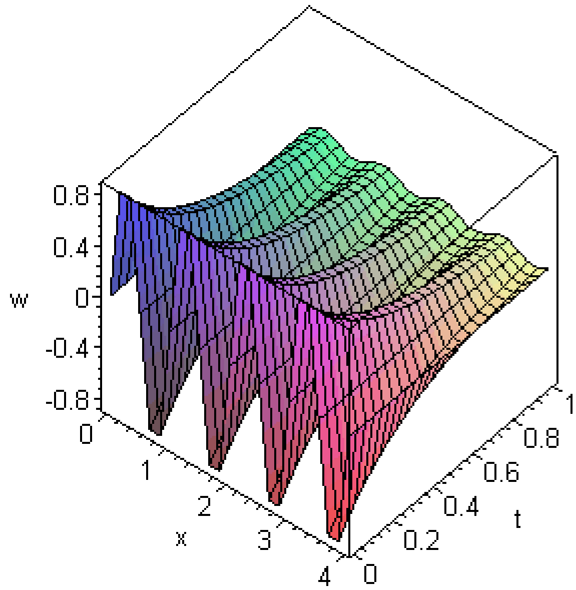
```
> with(plots):
```

Warning, the name changecoords has been redefined

```
> contourplot(w(x, t), x=0..4, t=0..1, labels=[x, t], contours=13, axes=boxed, filled=true,
  numpoints=1000);
```



```
> plot3d(w(x, t), x=0..4, t=0..1, labels=[x, t, w], axes=boxed, orientation=[-50, 45]);
```



What is happening as $t \to \infty$? Is this what you would expect? Why?

You Try It: Part I - Insulated Ends

We will next examine a very simple solution to the problem in which it is assumed that the ends of the rod are insulated so that no heat can escape. If the rod is of length 4, this is represented by the boundary conditions $w_x(0, t) = w_x(4, t) = 0$.

Why? Can you explain this in your own words?

Assume also that the initial temperature distribution within the rod is given by

$w(x, 0) = \cos(2\pi x)$. First, show that $w(x, t) = e^{(-3t)} \cos(2\pi x)$ is a solution to the

differential equation with these boundary and initial conditions. Also identify the constant *c*.

```
> restart;
```

```
> w:=(x, t)-> exp(-3*t)*cos(2*Pi*x):
  print(cat(`The change in temperature at the left end of the rod is `, diff(w(0, t), x))):
  print(cat(`The change in temperature at the right end of the rod is `, diff(w(4, t), x))):
  print(cat(`The initial temperature distribution is `, w(x, 0)));
  print(cat(`The second derivative of the temperature with respect to position is `, diff(diff(w
  x),x))):
```

```
wxx:=diff(diff(w(x, t), x),x):
print(cat(`The first derivative of the temperarure with respect to time is `, diff(w(x, t), t))):
wt:=diff(w(x, t), t):
c:=sqrt(wt/wxx):
print(`c= `, c, ` or `, evalf(c));
if wxx=1/c^2*wt then
print(cat(w(x, t), ` satisifes the differential equation.`));
else print(cat(`The equation is not satisfied.`));
fi;
```

The change in temperature at the left end of the rod is 0

The change in temperature at the right end of the rod is 0

The initial temperature distribution is $\|(\cos(2\pi x))$

The second derivative of the temperature with respect to position is $\|(-4\,\mathbf{e}^{(-3\,t)}\cos(2\pi x)\,\pi^2)$

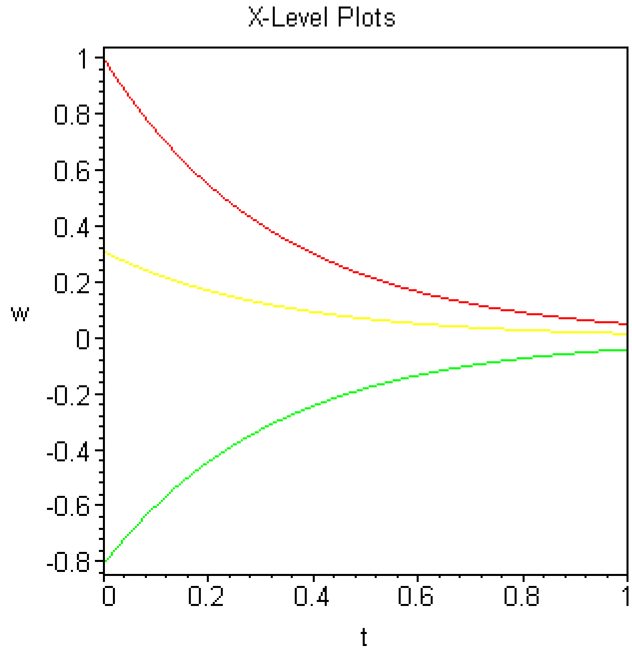
The first derivative of the temperarure with respect to time is $\|(-3\,\mathbf{e}^{(-3\,t)}\cos(2\pi x))$

$$c=,\ \frac{\sqrt{3}}{2\pi},\ or\ ,\ 0.2756644478$$

$\|(\mathbf{e}^{(-3\,t)}\cos(2\pi x))\|$ *satisifes the differential equation.*

Plot the temperature function at various positions along the rod as a function of time.

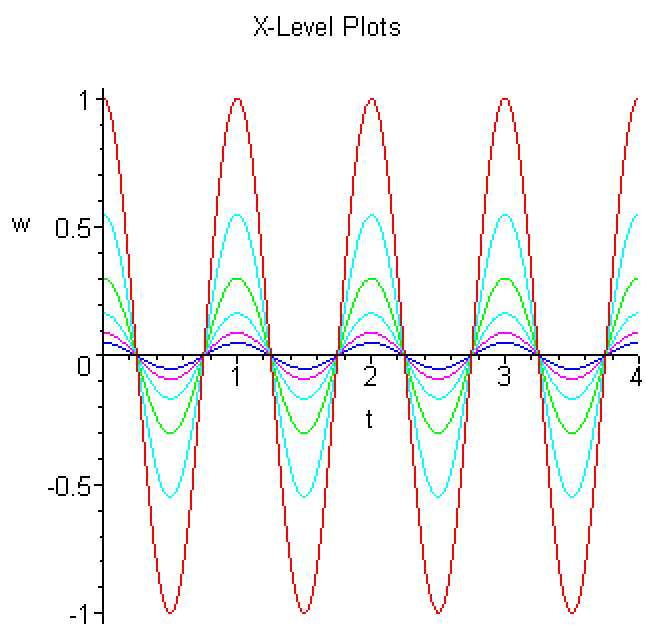
```
> plot([w(0, t), w(0.8, t), w(1.6, t), w(2.4, t), w(3.2, t), w(4, t)], t=0..1, color=[COLOR
(RGB,1,0,0),COLOR(RGB,1,1,0),COLOR(RGB,0,1,0),COLOR(RGB,0,1,1),COLOR
(RGB,1,0,1),COLOR(RGB,0,0,1)], labels=[t, w], title=`X-Level Plots`,axes=boxed);
```



Interpret these plots. What is happening to the temperature along different parts of the rod?

Plot the temperature function at various times as a function of the position on the rod.

```
> plot([w(x, 0), w(x, 0.2), w(x, 0.4), w(x, 0.6), w(x, 0.8), w(x, 1)], x=0..4, color=[COLOR(RGB,1,0,0),COLOR(RGB,0,1,1),COLOR(RGB,0,1,0),COLOR(RGB,0,1,1),COLOR(RGB,1,0,1),COLOR(RGB,0,0,1)], labels=[t, w], title=`X-Level Plots`);
```

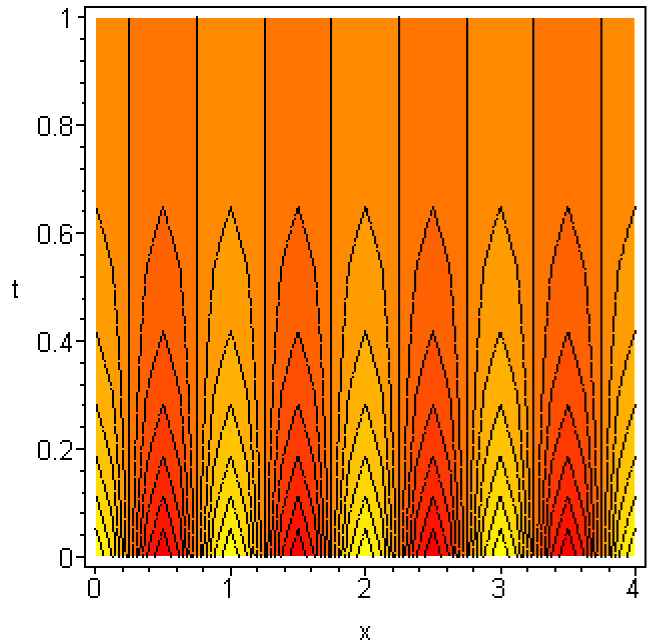


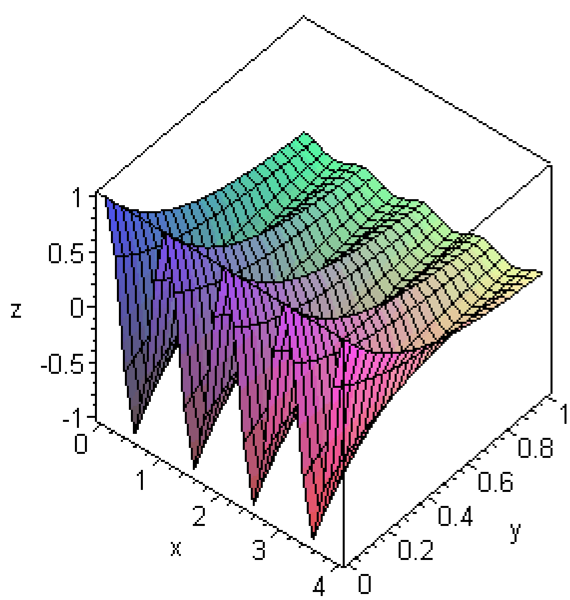
Interpret the results. What is happening to the temperature function as time goes on?

Plot the contours for the temperature function and the temperature function as a surface.

```
> with(plots):
  contourplot(w(x, t), x=0..4, t=0..1, labels=[x, t], contours=13, axes=boxed,
  filled=true,numpoints=1000);
  plot3d(w(x, t), x=0..4, t=0..1, labels=[x, t, w], axes=boxed, orientation=[-50, 45], labels=
  ["x","y","z"]);
```

Warning, the name changecoords has been redefined





What is happening as $t \to \infty$? Is this what you would expect? Why?

How does this differ from the temperature function in the rod with fixed ends?

Part II: Insulated Ends and Use of the Fourier Series (demonstration only)

If you have not learned the Fourier series at the end of Chapter 8 of your text, you should not do this part of the module.

Suppose that the rod of length 4 has insulated ends, that is, $w_x(0, t) = w_x(4, t)$ = 0. Assume a discontinuous initial temperature distribution that is 12 for the left $\frac{1}{4}$ of the rod and 0 elsewhere.

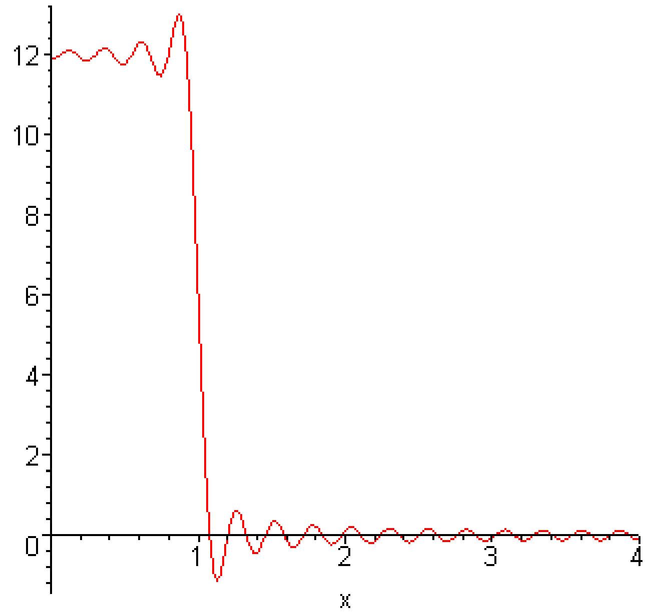
This discontinuous initial temperature function necessitates the use of the Fourier series in order to get a continuous approximation to this discontinuous initial state. = 0.

First, we determine the Fourier Series corresponding to the initial temperature state for the rod.

```
> restart;
```

```
> a:=n->(1/2)*int(12*cos(n*Pi*x/4), x=0..1):
  fourcos:=evalf((a(0)/2+sum(a(j)*cos(j*Pi*x/4), j=1..30))):
  print(cat(`The Fourier cosine series out to n = 30 is `, fourcos));
  plot(fourcos, x=0..4);
```

The Fourier cosine series out to $n = 30$ *is* ||$(3. + 5.401897895 \cos(0.7853981635\, x)$
$+ 3.819718633 \cos(1.570796327\, x) + 1.800632632 \cos(2.356194490\, x) - 1.080379579 \cos(3.92$
$- 1.273239544 \cos(4.712388981\, x) - 0.7716996991 \cos(5.497787144\, x)$
$+ 0.6002108770 \cos(7.068583472\, x) + 0.7639437266 \cos(7.853981635\, x)$
$+ 0.4910816268 \cos(8.639379798\, x) - 0.4155306073 \cos(10.21017613\, x)$
$- 0.5456740904 \cos(10.99557429\, x) - 0.3601265263 \cos(11.78097245\, x)$
$+ 0.3177586997 \cos(13.35176878\, x) + 0.4244131814 \cos(14.13716694\, x)$
$+ 0.2843104155 \cos(14.92256511\, x) - 0.2572332331 \cos(16.49336143\, x)$
$- 0.3472471485 \cos(17.27875960\, x) - 0.2348651258 \cos(18.06415776\, x)$
$+ 0.2160759158 \cos(19.63495409\, x) + 0.2938245103 \cos(20.42035225\, x)$
$+ 0.2000702924 \cos(21.20575041\, x) - 0.1862723412 \cos(22.77654674\, x)$
$- 0.2546479089 \cos(23.56194490\, x))$



The solution to the heat equation can now be written as

$$\frac{a(0)}{2}+\left(\sum_{n=1}^{\infty} a(n)\, e^{\left(-\left(\frac{n\,\pi}{4}\right)^{2} t\right)} \cos\left(\frac{n\,\pi\,x}{4}\right)\right)$$

and we will write write it out to $n = 30$.

```
> w1(x,t):=a(0)/2+add(a(j)*exp(-(j*Pi/4)^2*t)*cos(j*Pi*x/4),j = 1..30):
  evalf(%);
```

$$3. + 5.401897895\, e^{(-0.6168502752\, t)} \cos(0.7853981635\, x)$$
$$+ 3.819718633\, e^{(-2.467401101\, t)} \cos(1.570796327\, x)$$
$$+ 1.800632632\, e^{(-5.551652477\, t)} \cos(2.356194490\, x)$$
$$- 1.080379579\, e^{(-15.42125688\, t)} \cos(3.926990818\, x)$$
$$- 1.273239544\, e^{(-22.20660991\, t)} \cos(4.712388981\, x)$$
$$- 0.7716996991\, e^{(-30.22566349\, t)} \cos(5.497787144\, x)$$
$$+ 0.6002108770\, e^{(-49.96487230\, t)} \cos(7.068583472\, x)$$
$$+ 0.7639437266\, e^{(-61.68502752\, t)} \cos(7.853981635\, x)$$
$$+ 0.4910816268\, e^{(-74.63888331\, t)} \cos(8.639379798\, x)$$
$$- 0.4155306073\, e^{(-104.2476965\, t)} \cos(10.21017613\, x)$$

$$- 0.5456740904\, \mathbf{e}^{(-120.9026539\, t)} \cos(10.99557429\, x)$$
$$- 0.3601265263\, \mathbf{e}^{(-138.7913119\, t)} \cos(11.78097245\, x)$$
$$+ 0.3177586997\, \mathbf{e}^{(-178.2697295\, t)} \cos(13.35176878\, x)$$
$$+ 0.4244131814\, \mathbf{e}^{(-199.8594892\, t)} \cos(14.13716694\, x)$$
$$+ 0.2843104155\, \mathbf{e}^{(-222.6829494\, t)} \cos(14.92256511\, x)$$
$$- 0.2572332331\, \mathbf{e}^{(-272.0309714\, t)} \cos(16.49336143\, x)$$
$$- 0.3472471485\, \mathbf{e}^{(-298.5555332\, t)} \cos(17.27875960\, x)$$
$$- 0.2348651258\, \mathbf{e}^{(-326.3137956\, t)} \cos(18.06415776\, x)$$
$$+ 0.2160759158\, \mathbf{e}^{(-385.5314220\, t)} \cos(19.63495409\, x)$$
$$+ 0.2938245103\, \mathbf{e}^{(-416.9907861\, t)} \cos(20.42035225\, x)$$
$$+ 0.2000702924\, \mathbf{e}^{(-449.6838507\, t)} \cos(21.20575041\, x)$$
$$- 0.1862723412\, \mathbf{e}^{(-518.7710815\, t)} \cos(22.77654674\, x)$$
$$- 0.2546479089\, \mathbf{e}^{(-555.1652477\, t)} \cos(23.56194490\, x)$$

> **w:=unapply(w1(x,t),(x,t));**

$$w := (x, t) \to 3 + \frac{12\sqrt{2}\, \mathbf{e}^{\left(-\frac{1}{16}\pi^2 t\right)} \cos\left(\frac{1}{4}\pi x\right)}{\pi} + \frac{12\, \mathbf{e}^{\left(-\frac{1}{4}\pi^2 t\right)} \cos\left(\frac{1}{2}\pi x\right)}{\pi}$$
$$+ \frac{4\sqrt{2}\, \mathbf{e}^{\left(-\frac{9}{16}\pi^2 t\right)} \cos\left(\frac{3}{4}\pi x\right)}{\pi} - \frac{12}{5} \frac{\sqrt{2}\, \mathbf{e}^{\left(-\frac{25}{16}\pi^2 t\right)} \cos\left(\frac{5}{4}\pi x\right)}{\pi} - \frac{4\, \mathbf{e}^{\left(-\frac{9}{4}\pi^2 t\right)} \cos\left(\frac{3}{2}\pi x\right)}{\pi}$$
$$- \frac{12}{7} \frac{\sqrt{2}\, \mathbf{e}^{\left(-\frac{49}{16}\pi^2 t\right)} \cos\left(\frac{7}{4}\pi x\right)}{\pi} + \frac{4}{3} \frac{\sqrt{2}\, \mathbf{e}^{\left(-\frac{81}{16}\pi^2 t\right)} \cos\left(\frac{9}{4}\pi x\right)}{\pi} + \frac{12}{5} \frac{\mathbf{e}^{\left(-\frac{25}{4}\pi^2 t\right)} \cos\left(\frac{5}{2}\pi x\right.}{\pi}$$
$$+ \frac{12}{11} \frac{\sqrt{2}\, \mathbf{e}^{\left(-\frac{121}{16}\pi^2 t\right)} \cos\left(\frac{11}{4}\pi x\right)}{\pi} - \frac{12}{13} \frac{\sqrt{2}\, \mathbf{e}^{\left(-\frac{169}{16}\pi^2 t\right)} \cos\left(\frac{13}{4}\pi x\right)}{\pi} - \frac{12}{7} \frac{\mathbf{e}^{\left(-\frac{49}{4}\pi^2 t\right)} \cos(}{\pi}$$

$$-\frac{4}{5}\,\frac{\sqrt{2}\,e^{\left(-\frac{225}{16}\pi^2 t\right)}\cos\left(\frac{15}{4}\pi x\right)}{\pi}+\frac{12}{17}\,\frac{\sqrt{2}\,e^{\left(-\frac{289}{16}\pi^2 t\right)}\cos\left(\frac{17}{4}\pi x\right)}{\pi}+\frac{4}{3}\,\frac{e^{\left(-\frac{81}{4}\pi^2 t\right)}\cos(\frac{9}{2}}{\pi}$$

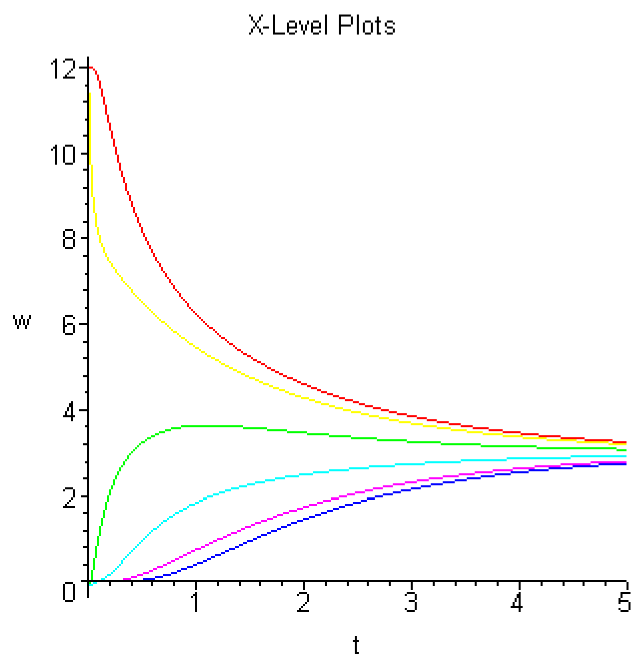
$$+\frac{12}{19}\,\frac{\sqrt{2}\,e^{\left(-\frac{361}{16}\pi^2 t\right)}\cos\left(\frac{19}{4}\pi x\right)}{\pi}-\frac{4}{7}\,\frac{\sqrt{2}\,e^{\left(-\frac{441}{16}\pi^2 t\right)}\cos\left(\frac{21}{4}\pi x\right)}{\pi}-\frac{12}{11}\,\frac{e^{\left(-\frac{121}{4}\pi^2 t\right)}\cos(}{\pi}$$

$$-\frac{12}{23}\,\frac{\sqrt{2}\,e^{\left(-\frac{529}{16}\pi^2 t\right)}\cos\left(\frac{23}{4}\pi x\right)}{\pi}+\frac{12}{25}\,\frac{\sqrt{2}\,e^{\left(-\frac{625}{16}\pi^2 t\right)}\cos\left(\frac{25}{4}\pi x\right)}{\pi}+\frac{12}{13}\,\frac{e^{\left(-\frac{169}{4}\pi^2 t\right)}\mathrm{co}}{\pi}$$

$$+\frac{4}{9}\,\frac{\sqrt{2}\,e^{\left(-\frac{729}{16}\pi^2 t\right)}\cos\left(\frac{27}{4}\pi x\right)}{\pi}-\frac{12}{29}\,\frac{\sqrt{2}\,e^{\left(-\frac{841}{16}\pi^2 t\right)}\cos\left(\frac{29}{4}\pi x\right)}{\pi}-\frac{4}{5}\,\frac{e^{\left(-\frac{225}{4}\pi^2 t\right)}\cos(}{\pi}$$

Plot the temperature function at various positions along the rod as a function of time.

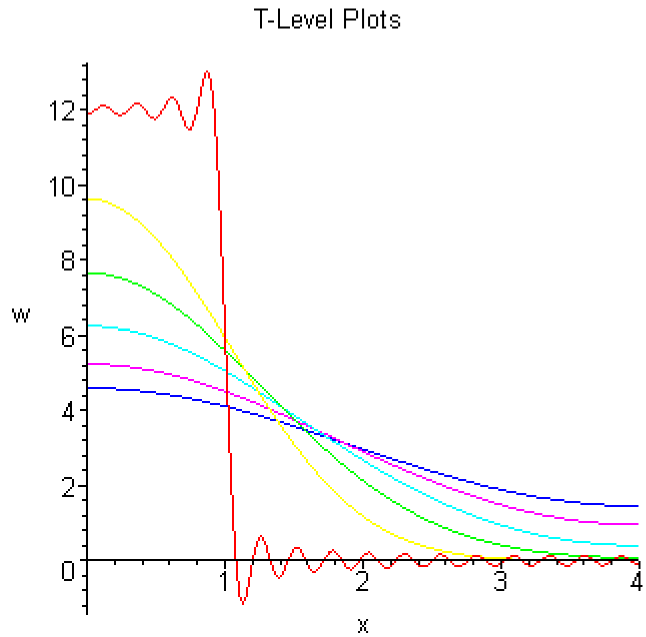
```
> xlevelplots:=plot([w(0, t), w(0.8, t), w(1.6, t), w(2.4, t), w(3.2, t), w(4, t)], t=0..5, color=[COL
(RGB,1,0,0),COLOR(RGB,1,1,0),COLOR(RGB,0,1,0),COLOR(RGB,0,1,1),COLOR
(RGB,1,0,1),COLOR(RGB,0,0,1)], labels=[t, w], title=`X-Level Plots`):
print(xlevelplots);
```



Interpret these plots. What is happening to the temperature along different parts of the rod?

Plot the temperature function at various times as a function of the position on the rod. Interpret the results.

```
> tlevelplots:=plot([w(x, 0), w(x, 0.3), w(x, .6), w(x, 1), w(x, 1.5), w(x, 2)], x=0..4, color=[COL(
(RGB,1,0,0),COLOR(RGB,1,1,0),COLOR(RGB,0,1,0),COLOR(RGB,0,1,1),COLOR
(RGB,1,0,1),COLOR(RGB,0,0,1)], labels=[x, w], title=`T-Level Plots`):
print(tlevelplots);
```

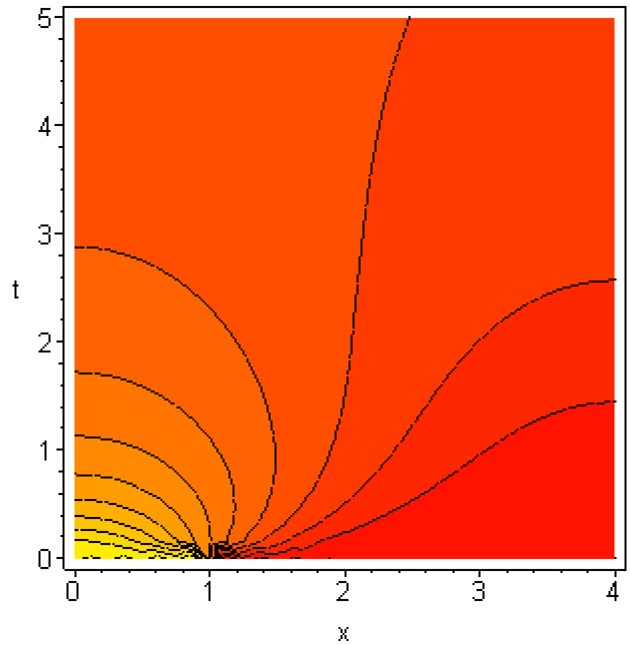


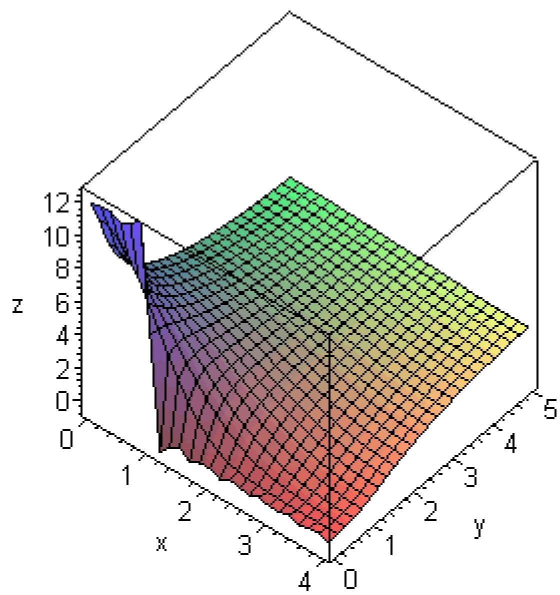
Interpret the results. What is happening to the temperature function as time goes on?

Plot the contours for the temperature function and the temperature function as a surface.

```
> with(plots):
contourplot(w(x, t), x=0..4, t=0..5, labels=[x, t], contours=13, axes=boxed,
filled=true,numpoints=1000);
plot3d(w(x, t), x=0..4, t=0..5, labels=[x, t, "w"], axes=boxed, orientation=[-50, 45], labels=
["x","y","z"]);
```

Warning, the name changecoords has been redefined





>

What is happening as $t \to \infty$? Is this what you would expect? Why?