Take It to the Limit

Note: You may notice differences between this Maple worksheet and the equivalent Mathematica notebook. These differences were introduced to preserve the content of these modules and were necessary because of major functional differences between Maple and Mathematica.

In executing this module, you can expect to see an error in the evaluation. This will happen whenever you try to divide by 0, as we do in certain evaluations.

Introduction

OBJECTIVE: To interpret the limit concept graphically and numerically.

In this module, you will explore some limits by graphing the functions involved and by creating tables of values for the functions. You will explore, in some detail, functions that have indeterminate forms at the limit point, and you will come to appreciate more fully the importance of formal proofs in mathematics.

Technology Guidelines

NOTE: If you have just finished a worksheet, **restart** *Maple* before executing a new worksheet.
TO OPEN SECTIONS,
Click on the **PLUS** sign at the left hand side of the screen *or* select **Expand All Sections** from the **View** drop down menu.

TO STOP AN EXECUTION
Click on **STOP** button from the toolbar.

ORDER OF EXECUTION
Execute commands in the order given. Do not skip any *Maple* Input lines within a given worksheet

Alternatively, you can execute the entire worksheet by selecting the **Execute Worksheet** command from the **Edit** drop down menu.

SAVING WORKSHEETS.

You can save anytime to any directory you choose, and it is wise to save often.
EXPERIENCING MAJOR PROBLEMS
Save if appropriate and then shut down *Maple* and start it up again.

Part I: What is 0/0 ?

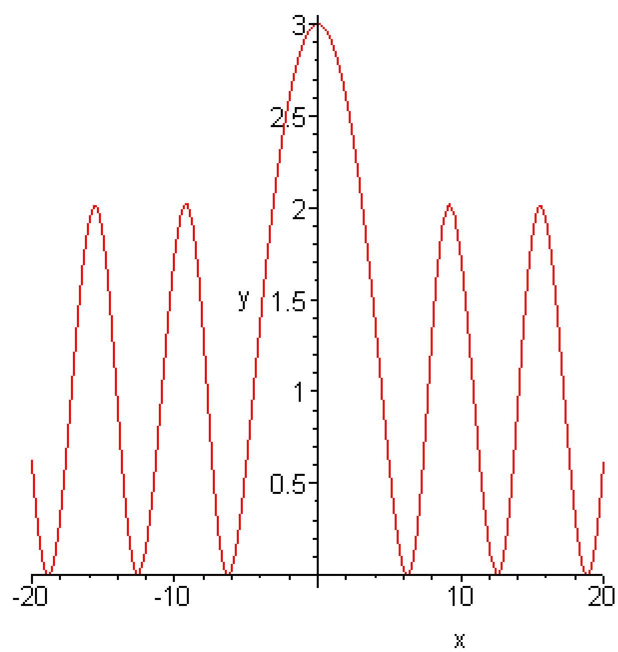
Section 2.3, Exercise 64

When we try to evaluate a function like $\frac{x\,(1-\cos(x))}{x-\sin(x)}$ at x=0, we get $\frac{0}{0}$, but what does this mean? The answer to the question "What is 0/0?" depends on whether the numerator or the denominator of the rational function we are trying to evaluate approaches 0 faster. This function is not defined at x=0. because its denominator is 0 there; however, the numerator is also 0 at x=0. To better understand the situation, let's investigate what happens to the values of $\frac{x\,(1-\cos(x))}{x-\sin(x)}$ when x is very near 0. We begin by graphing the function.

```
> restart:
```

```
> f:=x-> x*(1-cos(x))/(x-sin(x));
  plot(f(x), x=-20..20, labels=[x,y]);
```

$$f := x \rightarrow \frac{x\,(1-\cos(x))}{x-\sin(x)}$$



The graph seems to indicate that this function is approaching 3 as x approaches 0. To see that this is the case, let's compute some values of the function for x near 0. The **matrix** command provides a way to do this, and we use it to create a list of entries, {x, $f(x)$, $f(x)$-3} for values ranging from -1 to 1 in increments of 0.1. Don't be surprised by the error messages that are produced when you execute the next command; they are to remind you that $f(x)$ is not defined at x=0.

```
> Digits:=8:
  count:=0:
  for i from -1.0 to 0 by 0.1 do
    mm[count]:=([i,f(i), f(i)-3]);
    count:=count+1:
  od:
```

```
matrix([[`x`, `f(x)`, `f(x)-3`], seq(mm[i], i=1..count-1)]);

Digits:=8:
count:=0:
for i from 1.0 to 0 by -0.1 do
  mm[count]:=([i,f(i), f(i)-3]);
  count:=count+1:
od:
matrix([[`x`, `f(x)`, `f(x)-3`], seq(mm[i], i=1..count-1)]);
```

x	$f(x)$	$f(x)-3$
-0.9	2.9188481	-0.0811519
-0.8	2.9359045	-0.0640955
-0.7	2.9509439	-0.0490561
-0.6	2.9639692	-0.0360308
-0.5	2.9749855	-0.0250145
-0.4	2.9839934	-0.0160066
-0.3	2.9910002	-0.0089998
-0.2	2.9959975	-0.0040025
-0.1	2.9990035	-0.0009965
0.	Float(undefined)	Float(undefined)

x	$f(x)$	$f(x)-3$
0.9	2.9188481	-0.0811519
0.8	2.9359045	-0.0640955
0.7	2.9509439	-0.0490561
0.6	2.9639692	-0.0360308
0.5	2.9749855	-0.0250145
0.4	2.9839934	-0.0160066
0.3	2.9910002	-0.0089998
0.2	2.9959975	-0.0040025
0.1	2.9990035	-0.0009965
0.	Float(undefined)	Float(undefined)

How close to 0 must I hold x in order to guarantee that $f(x)$ stays within 1/100th of 3? You will explore that in the You Try It: Part I section.

▪ Continuity Issues: Section 2.6

You might notice from the graph of $f(x)$ that it appears to be continuous. For a function to be continuous, the limit as x approaches 0 must exist and this condition seems to be met. However, continuity requires that the function be defined when x is 0, and here we have a problem. If we evaluate $f(0)$, the result is an indeterminate form (0/0 is such a form).

```
> f(0);
```

Error, (in f) numeric exception: division by zero

Is f(x) continuous at x=0? why?

The graph of f(x) is deceiving. Mathematically, there is a gaping hole in the graph at the point (0,3) even though we can't see it on the *Maple* graph of $f(x)$. As a result, we might be falsely led to believe that $f(x)$ is continuous at x=0. Now consider the function, $g(x) = \frac{x(1-\cos(x))}{x-\sin(x)}$ if $x \neq 0$ and $g(x) = 3$ if x=0. What is $g(0)$? What is

$\lim_{x \to 0} g(x)$? Is this function continuous at x=0?

The discontinuity in $f(x)$ at x=0 is said to be "removable" since we can remove it by forming a new continuous function, $g(x)$, that is identical to $f(x)$ when $x \neq 0$ and is equal to 3 when x=0.

For further exploration of continuous and discontinuous functions, refer to "Continuous and Discontinuous Curves," a JAVA applet included in this supplement.

You Try It: Part I

▪ Take the Epsilon Challenge

We have a challenge for you. Here it is.

Determine how close x must be to 0 so that the values of $f(x)$ will be within a specified distance from 3. More specifically, we challenge you with a small positive number, let's call it ε , and ask you try to determine a value for δ so that when $-\delta < x < \delta$ with $x \neq 0$, the values of $f(x)$ will satisfy the inequality, $|f(x) - 3| < \varepsilon$, that is, the distance between the value of $f(x)$ and 3 will be less than ε . For example, if $\varepsilon = .1e\text{-}1$ the table of values in the cell above shows that $-.1 < f(x) - 3 < .1$ is not satisfied when $\delta = 1.$ You need to find a smaller δ that will keep x closer to 0 so that you can beat our

ε challenge.

To help you with this, we include a command in the next cell that generates a list of ordered pairs $[x, |f(x) - 3|]$ so that you can look at the magnitude of the differences between the values of *f*(*x*) and 3 to see if the δ you picked meets our ε challenge. In addition, we added to each ordered pair the result of a test to see if a is within a distance ε of 3. We start with $\varepsilon = .1e\text{-}1$ and a value for δ that is too big. You should try different values for δ, re-evaluating the commands in the cell each time, until you find a δ that works. After you find a δ that works for $\varepsilon = .1e\text{-}1$, try to find δ -values that work for

$\varepsilon = .1e\text{-}3, .1e\text{-}6, 10^{(-10)}$, ...

```
> Digits:=25:
  delta:=0.0025:
  epsilon:=0.01:
  L:=3:
  count:=0:
  for i from -delta to delta by delta/10 do
    if (evalf(abs(f(i)-L)) < epsilon)  then result:=true: else result:=false : fi:
    mm[count]:=([i, abs(f(i)-L), result]);
    count:=count+1
  od:
  matrix([[`x` , `|f(x)-L|` , `|f(x)-L|< epsilon`], seq(mm[i], i=1..count-1)]);
```

x	$\lvert f(x)-L \rvert$	$\lvert f(x)-L \rvert <$ *epsilon*
-0.002250000000000000000000000	$0.506250006102206238\ 10^{-6}$	*true*
-0.002000000000000000000000000	$0.400000003808443143\ 10^{-6}$	*true*
-0.001750000000000000000000000	$0.306250002232958493\ 10^{-6}$	*true*
-0.001500000000000000000000000	$0.225000001205032113\ 10^{-6}$	*true*
-0.001250000000000000000000000	$0.156250000578870841\ 10^{-6}$	*true*
-0.001000000000000000000000000	$0.100000000237399893\ 10^{-6}$	*true*
-0.0007500000000000000000000000	$0.56250000075897898\ 10^{-7}$	*true*
-0.0005000000000000000000000000	$0.25000000013299998\ 10^{-7}$	*true*
-0.0002500000000000000000000000	$0.6249999945931250\ 10^{-8}$	*true*
0.	Float(undefined)	*false*
0.0002500000000000000000000000	$0.6249999945931250\ 10^{-8}$	*true*
0.0005000000000000000000000000	$0.25000000013299998\ 10^{-7}$	*true*
0.0007500000000000000000000000	$0.56250000075897898\ 10^{-7}$	*true*
0.001000000000000000000000000	$0.100000000237399893\ 10^{-6}$	*true*
0.001250000000000000000000000	$0.156250000578870841\ 10^{-6}$	*true*
0.001500000000000000000000000	$0.225000001205032113\ 10^{-6}$	*true*
0.001750000000000000000000000	$0.306250002232958493\ 10^{-6}$	*true*
0.002000000000000000000000000	$0.400000003808443143\ 10^{-6}$	*true*
0.002250000000000000000000000	$0.506250006102206238\ 10^{-6}$	*true*
0.002500000000000000000000000	$0.625000009300793842\ 10^{-6}$	*true*

The formal definition of the limit says that if you are able meet our ε challenge, no matter how small we make our ε and if you are able to prove that you can do, then you can say unequivocally that $\lim_{x \to 0} f(x) = 3$. We aren't going to ask you to prove this limit, but your instructor may ask you to prove some simpler ones. One thing you can do, however, is use *Maple's* **limit**( ) command to see if it verifies our conjecture that $\lim_{x \to 0} f(x) = 3$.

```
> limit( f(x), x = 0);
```

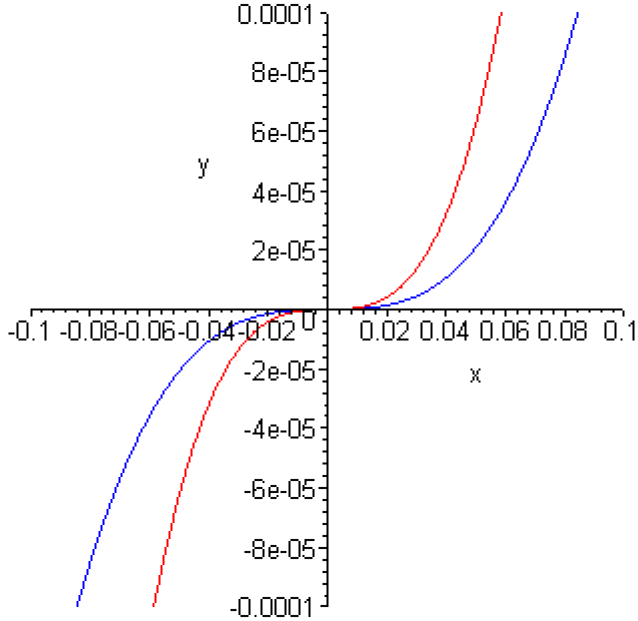
$$3$$

▪ What's Up? (and Down?)

We can get more insight into what is happening to the function $\frac{x\,(1-\cos(x))}{x-\sin(x)}$ as x gets close to 0 by looking at the numerator and denominator of the function. In the following cell, we plot the numerator and denominator as separate functions of x.

We set the viewing window with the range in the option for the **plot**( ) command so that the relation between the values of the numerator and denominator can be easily seen. You can zoom in closer by letting *xb* and *yb* get smaller and smaller. Your bounds for *y* will have to be much, much smaller than your bounds for *x*; otherwise you will not be able to see your graphs.

```
> num:= x*(1-cos(x)):
  den:=x-sin(x):
  xb:= 0.1:
  yb:= 10^(-4):
  plot({num, den}, x=-xb..xb, y=-yb..yb, color=[blue,red]);
```



Which curve do you think is the graph of the numerator function, and which is the denominator? Why? See if you can differentiate between the two curves as *x* gets closer to 0.

Use the graphs above to find an approximate numeric relation between the value of the numerator and the value of the denominator for *x* values that are close to 0. Based upon the relationship you find, explain why the value of $\lim_{x \to 0} \frac{x\,(1-\cos(x))}{x-\sin(x)}$ is not surprising.

Try to generate a table with values of **x, num, den,** and $\frac{num}{den}$ in each row, for values of *x* close to 0. Comment on what is happening to the values of **num, den** and $\frac{num}{den}$ as *x* gets closer to 0.

- **Try another $\frac{0}{0}$ expression**

Find the limit as $x \to 1$ of the function $\frac{\log(x)}{1-2^{x-1}}$

Start by trying to evaluate the function at x =1.

```
> f:=x->log(x)/(1-2^(x-1));
```

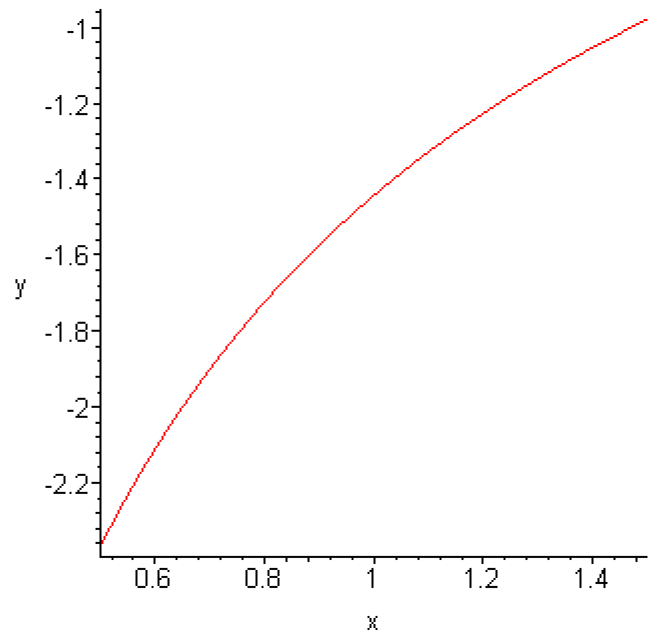
```
f(1);
```

$$f := x \rightarrow \frac{\log(x)}{1 - 2^{(x-1)}}$$

Error, (in f) numeric exception: division by zero

We had better graph the function to see what is happening.

```
> plot(f(x), x=0.5..1.5, labels=["x","y"]);
```



How can you determine the precise limit? Try looking at some values of the function for x near 1.

```
> count:=0:
  for i from 0.9 to 1.1 by 0.01 do
   mm[count]:=([i, (f(i))]);
   count:=count+1:
  od:
  matrix([[`x`, `f(x)`], seq(mm[i], i=1..count-1)]);
```

x	$f(x)$
0.91	-1.5594405870891846184448892
0.92	-1.5457540110437865358591170
0.93	-1.5322556023319863262900222
0.94	-1.5189411077704425949105688
0.95	-1.5058064139470605556116844
0.96	-1.4928475412256305432696339
0.97	-1.4800606380663464984610009
0.98	-1.4674419756425232522503331
0.99	-1.4549879427352435440577780
1.00	Float(undefined)
1.01	-1.4305598798122981649782210
1.02	-1.4185791730093100759473380
1.03	-1.4067497336276310465371179
1.04	-1.3950684705106873223911119
1.05	-1.3835323844421545528141138
1.06	-1.3721385645715672464282209
1.07	-1.3608841850107420203662214
1.08	-1.3497665015913542597043310
1.09	-1.3387828487746374563096009
1.10	-1.3279306367047577700817731

To investigate this limit, try some explorations like those in Part I.

Part II: What is 0^0 ?

How does a function like $|x|^x$ behave as x gets closer and closer to 0? The answer is not obvious, since numbers to the 0 power got to 1, but 0 to any power is 0. We will start by trying the **limit** command.

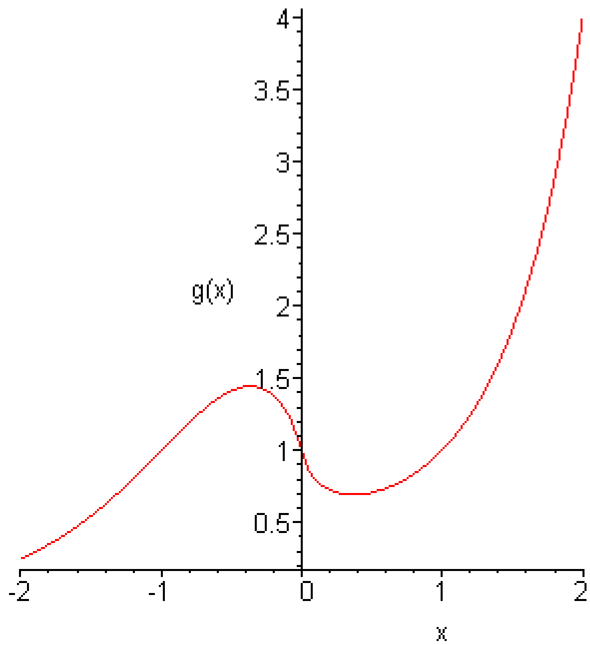
```
> restart;
  g:=x->abs(x)^x;
  limit(g(x), x=0);
```

$$g := x \to |x|^x$$

$$1$$

Is this correct? We might look at the graph of the funciton. Note what happens when *x* is 0 on the graph.

```
> plot(g(x), x=-2..2, labels=[x,"g(x)"]);
```



▪ Continuity Issues: Section 2.6

If we ask for *g*(0), the result should indicate that it is an indeterminate form (0^0 is such a form).

```
> g(0);
```

$$1$$

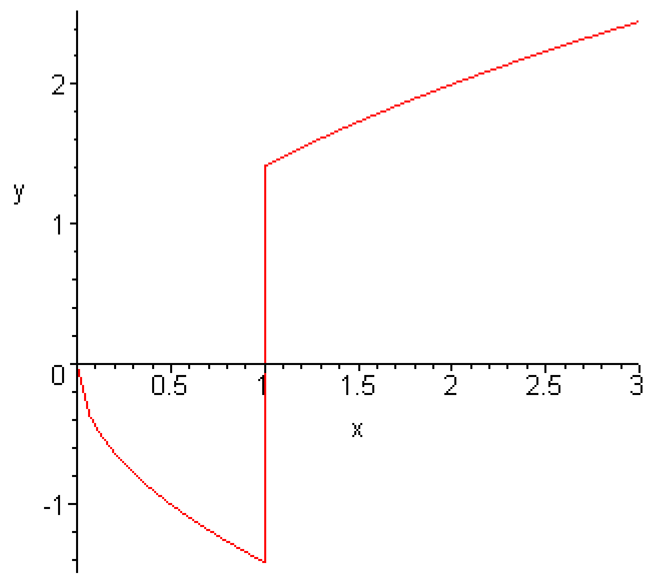
The graph of *g*(*x*) appears to be continuous at *x*=0. Is it? Why?

Part III: One sided Limits

Section 2.4, Exercise #18

Consider the function $\frac{\sqrt{2x}\,(x-1)}{|x-1|}$. This function appears to have problems at $x = 1$. Let's begin by looking at the graph.

```
> h:=x-> sqrt(2*x)*(x-1)/abs(x-1):
  plot(h(x), x=0..3, labels=[x,y]);
```



The vertical line at $x = 1$ really shouldn't be there. You can remove the line by setting the print options **discont=true.** The lower piece is not really connected to the upper piece. *Maple* graphs a function by plotting a series of points on the funciton and then connects them with straight lines, sometimes when they really shouldn't be connected.

What is happening at $x = 1$? is $h(x)$ defined at $x = 1$?

```
> h(1);
```

Error, (in h) numeric exception: division by zero

Lets' try the **limit()** command. Becasue of the input at x=1, we will use on-sided limits. The option **left** indicates that we are approaching the value specified from values that are smaller , whereas **right** indicates that we are approaching the value specified from values that are larger.

```
> limit(sqrt(2*x)*(x-1)/(-(x-1)), x=1, left);
```

$$-\sqrt{2}$$

```
> limit(sqrt(2*x)*(x-1)/((x-1)), x=1, right);
```

$$\sqrt{2}$$

This result verifies what is shown in the graph. That is $\lim_{x \to 1^{-1}} h(x) = -\sqrt{2}$ and $\lim_{x \to 1^{+}} h(x) = \sqrt{2}$. We can check this out by looking at some values of the function $x = 1$ and comparing these values with $\sqrt{2}$.

```
> count:=1:
  for i from 0.999 to 1.001 by 0.0001 do
    mm[count]:=([i, (h(i))]);
    count:=count+1:
  od:
  matrix([[`x`, `h(x)`], seq(mm[i], i=1..count-1)]);
```

x	$h(x)$
0.999	-1.413506279
0.9991	-1.413577023
0.9992	-1.413647764
0.9993	-1.413718501
0.9994	-1.413789235
0.9995	-1.413859965
0.9996	-1.413930691
0.9997	-1.414001414
0.9998	-1.414072134
0.9999	-1.414142850
1.0000	Float(undefined)
1.0001	1.414284271
1.0002	1.414354977
1.0003	1.414425678
1.0004	1.414496377
1.0005	1.414567072
1.0006	1.414637763
1.0007	1.414708451
1.0008	1.414779135
1.0009	1.414849816
1.0010	1.414920492

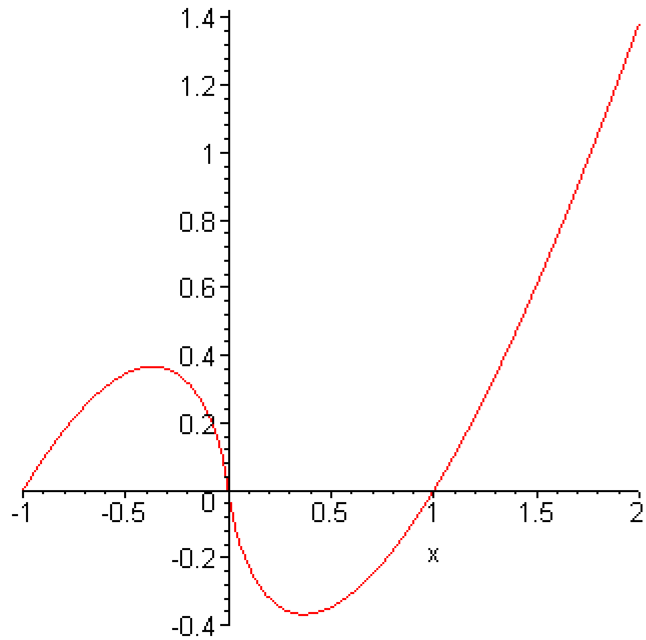
The value of $\sqrt{2}$ is 1.41421

You Try It: Part III

Find the limit as $x \to 0^+$ of $x \log(x)$.

```
> g:=x->x*log(abs(x)):
  g(0);
  plot(g(x), x=-1..2);
```

Error, (in ln) numeric exception: division by zero



What do you think is happening around $x = 0$? How do we know for sure that there is not a jump discontinuity at $x = 0$? Maybe, if we zoom in close enough to $x = 0$, we will see a jump. How can we tell for sure whether ther is or is not a jump at $x = 0$?

```
> limit(x*log(x), x=1, right);
```

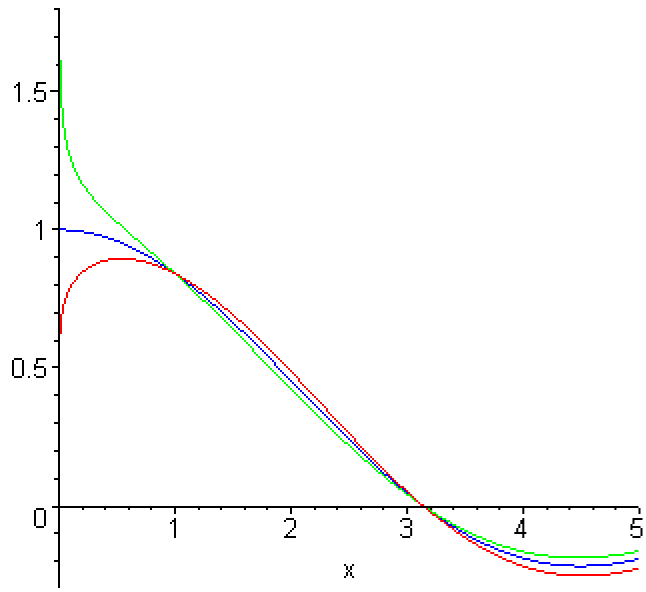
$$0$$

Part IV: What a Difference a Power Makes!

In your coursework, you have undoubtedly studied the limit of the $\frac{\sin(x)}{x}$ as $x \to 0$ and have found that its value is 1. What happens to the limit if we change the power of x in the denominator to values that are near but not equal to 1? We begin by graphing three of these functions and see what the graphs suggest. The function $\frac{\sin(x)}{x}$ see what the graphs suggest. The function $\frac{\sin(x)}{x}$ will be graphed in red, $\frac{\sin(x)}{x^{.9}}$ in green, and $\frac{\sin(x)}{x^{1.1}}$ in blue. Because the roots of negative numbers are not always real numbers, we use a right-hand limit, approaching 0 only from the right.

```
> restart:
  f1:=sin(x)/x:
  f2:=sin(x)/x^(0.9):
```

```
f3:=sin(x)/x^(1.1):
plot({f1, f2, f3}, x=0..5, color=[red,green, blue]);
```



Based on this graph, what would you guess are the limits of the three functions?

Now let's evaluate each of the functions at $x = 10^{(-100)}$.

```
> evalf(subs(x=10^(-100), f1));
  evalf(subs(x=10^(-100), f2));
  evalf(subs(x=10^(-100), f3));
```

$$1.000000000$$

$$0.1000000000\ 10^{-9}$$

$$0.1000000000\ 10^{11}$$

Do these results confirm or contradict the conjectures you made for the values of the three limits?

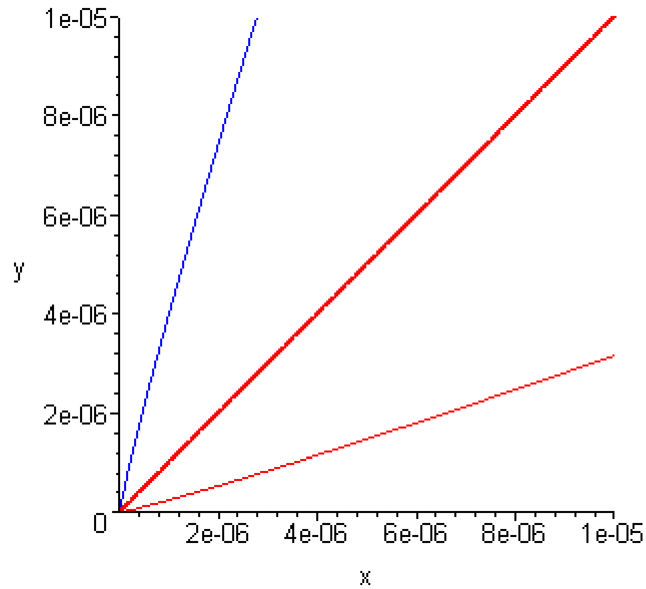
You Try It: Part IV

Proofs are Important!

- **Closely examine the $\frac{\sin(x)}{x^n}$ problem.**

The following commands will result in the sine functinon being plotted in the thicker form, *x* in red, $x^{.9}$ in green and $x^{1.1}$ in blue.

```
> num:=sin(x):
  den1:=x:
  den2:=x^(.9):
  den3:=x^(1.1):
  b:=10^(-5):
  plot({num, den1, den2, den3}, x=0..b , y=0..b, thickness=[3,1,1,1], color=[red, green, blue]);
```




```
> ?

>
```

What's going on here? Without being more careful, we might conclude from the graphs above that in all three cases the ratio of **num** to **denom** is a constant and that each limit should be approaching some finite number. To see what is really happening, you need to look at several graphs of the numerator and denominator functions, each one zoomed by a factor of $\frac{1}{100}$. Go back to the preceding cell and re-execute the commands with the exponent of **b =** -7, -9, -11, . . . What does this tell you about the ratio of **num** to **denom** in each of the three limits as *x* approaches 0?

Wait a minute! How do we know for sure that the pattern illustrated in the graphs above will continue if we zoom in even closer. Maybe things will "settle down" and the ratios will, in fact, approach constant values. What about the function $\frac{x\,(1-\cos(x))}{x-\sin(x)}$ from Part I? Maybe, if we zoom in close enough to 0, the graph of **num** to **denom** will start to look like the correspondin graphs for one of the three functions considered in this Part. How can we be sure of the situation? The answer is in the formal definition of the limit and proof. As mathematicians, we are not fully satisfied until we can prove our conjectures about the behavior of the functions as *x* approaches 0. Then, there is no doubt.