Bouncing Ball

Note: You may notice differences between this Maple worksheet and the equivalent Mathematica notebook. These differences were introduced to preserve the content of these modules and were necessary because of major functional differences between Maple and Mathematica.

Introduction

OBJECTIVE: To investigate an application using geometric series and the capabilities of technology.

In this module, you will use geometric series to investigate the behavior of a bouncing ball. Properties such as distance traveled and time bouncing will be determined. Enjoy your inquiry

Technology Guidelines

NOTE: If you have just finished a worksheet, **restart** *Maple* before executing a new worksheet.
TO OPEN SECTIONS,
Click on the **PLUS** sign at the left hand side of the screen *or* select **Expand All Sections** from the **View** drop down menu.

TO STOP AN EXECUTION
Click on **STOP** button from the toolbar.

ORDER OF EXECUTION
Execute commands in the order given. Do not skip any *Maple* Input lines within a given worksheet

Alternatively, you can execute the entire worksheet by selecting the **Execute Worksheet** command from the **Edit** drop down menu.

SAVING WORKSHEETS.

You can save anytime to any directory you choose, and it is wise to save often.
EXPERIENCING MAJOR PROBLEMS
Save if appropriate, and then shut down *Maple* and start it up again.

Part I: How Far Does the Ball Travel?

Section 11.2, Example 3

Sequences and series can help determine the behavior of a bouncing ball. Suppose you drop a ball from h meters above a hard, flat surface . Each time the ball bounces, it rebounds by a factor of r times its previous height. The value of r depends on the elasticity of the ball and the physical properties of the surface. The terminology usually used for r is the coefficient of restitution. If the ball is a typical ball that we use for games (e.g., basketball, soccer ball), then $r<1$. If

$r>1$, the ball violates basic physical principles and acts like a ball made of flubber.

Now let's see what we can determine mathematically about the behavior of a bouncing ball.

First, we drop the ball from 10 meters and see that it rebounds 3.7 meters on the first bounce. Therefore, $h = 10$ and $r = 0.37$. What is the height of the 10th bounce? Since each bounce produces a height of r times the previous, the first bounce is hr and the second has height hr^2.

The height of the 10th bounce is $h\,r^{10}$. For these values, we find the height of the 10th bounce by evaluating $10*(.37^{10})$.

```
> restart:
> 10*.37^10;
```

0.0004808584372

This really isn't very high. We probably couldn't see the 10th bounce of this ball.

How far does the ball travel during those 10 bounces? The distance traveled is made up of 3 parts:1) the first drop to the surface (10 meters for our ball), 2) the 9 bounces up and back down (each bounce adds $2(10)\,.37^k$ meters to the distance for k=1 to 9), and 3) the final (or 10th) bounce upward:

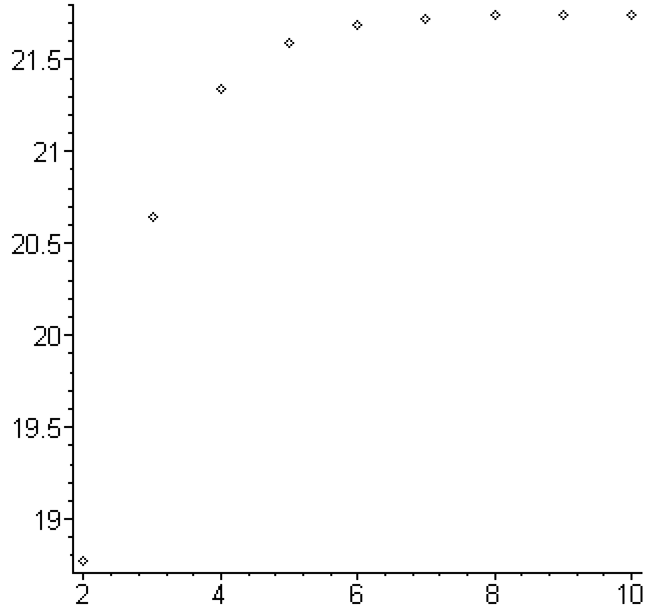
$10*(.37^{10})$. (the value we determined). If we put this all together in a summation, we find the total distance traveled as: $10 + 2\left(\sum_{k=1}^{9} 10\ .37^k\right) + 10\ .37^{10}$. Let's compute this value.

```
> 10+2*sum(10*.37^k, k=1..9)+10*.37^10;
```

21.74498607

So we see the ball has traveled quite a distance, 21.745 meters, during the 10 bounces. Let's plot how far the ball has traveled after each bounce. We can do this by writing the function $g(n) = 10 + 2\left(\sum_{k=1}^{n-1} 10\ .37^k\right) + 10\ .37^n$, where $g(n)$ is the distance traveled on bounce n. Let's enter $g(n)$ and plot the values of $g(n)$ for n =2 to 20.

```
> data:=[seq([i,subs(n=i, 10+2*sum(10*.37^k, k=1..n-1)+10*.37^n)], i=2..10)]:
  plots[pointplot](data);
```



For generic values of h, r and n bounces, the distance fucntion becomes

$$d(h, r, n) = h + 2\left(\sum_{k=1}^{n-1} h\,r^k\right) + h\,r^n$$

You Try It: Part I

How far does a ball travel in 6 bounces when it is dropped from 15 meters and the ball bounces 8 meters high on the first bounce?

What is the height of the 20th bounce if $h = 30$ and $r = .93$?

You want the 5th bounce to be 3 meters high for a ball and surface with $r = .75$, how high do you need to drop the ball?

How far does a ball travel in 12 bounces when it is dropped from 100 meters and the ball bounces 42 meters high on the first bounce?

Part II: What Happens for an Infinite Number of Bounces?

Section 11.2, Exercise 73

Let's allow our ball to continue to bounce so that we can determine how far the ball travels after an infinite number of bounces. Our summation becomes an infinite series. If we put this all together, we find the total distance traveled for a ball dropped from height *h* and rebounding by a factor *r* as: $S(h, r) = h + 2\left(\sum_{k=1}^{\infty} h\, r^k\right)$. Let's see what happens if we return to our problem with $h = 10$ and $r = 0.37$. How far does the ball travel with an infinite number of bounces? We compute s(*10, 0.37*) *by* $10 + 2\left(\sum_{k=1}^{\infty} 10.37^k\right)$.

```
> 10+2*sum(10*0.37^k, k=1..infinity);
```

21.74603175

Amazing! The result is just a little larger than the distance traveled after 10 bounces. Does this series converge? Yes, it's a geometric series with $r < 1$. That's how its value is computed. Let's do a couple more series calculations and then figure out the formula for this series.

What happens if we double the starting height? *Let* $h = 20$ instead of 10.

```
> 20+2*sum(20*0.37^k, k=1..infinity);
```

43.49206350

The value of the distance is doubled. Should we have expected that? Now let's vary the rebound factor *r*. Let's leave $h = 10$, but double *r* to $r = 0.74$. What will happen in this case?

```
> 10+2*sum(10*0.74^k, k=1..infinity);
```

66.92307692

This time the result has not doubled; it's more than tripled. We can do two things to continue the analysis of various scenarios for *h* and *r*. We can enter the generic series function

$S(h, r) = h + 2\left(\sum_{k=1}^{\infty} h\, r^k\right)$, substituting values as needed, or we can find an explicit formula

for *S*(*h*,*r*) using the result for the values of a geometric series given in Section 8.3. Let's find the explicit formula first. Given that $\sum_{k=1}^{\infty} a\, r^{(k-1)} = \frac{a}{1-r}$ (for $|r| < 1$), multiplying by *r* produces $\sum_{k=1}^{\infty} a\, r^{(k-1)} = \frac{a\, r}{1-r}$ (for |*r*|<*1*). Therefore, $S(h, r) = h + 2\left(\sum_{k=1}^{\infty} h\, r^k\right)$ =

$h + \frac{2\, h\, r}{1-r} = \frac{h\,(1+r)}{1-r}$.

Let's check this explicit formula with *h =10* and *r =.74.*

```
> 10*(1+.74)/(1-.74);
```

66.92307692

This agrees with our previous result. We now use both methods to investigate the distance traveled for a bouncing ball with *h =10* and *r =0.185*.

```
> s:=(h,r)->h+2*sum(h*r^k, k=1..infinity):
  s(10,0.185);
```

14.53987730

```
> 10*(1+0.185)/(1-.185);
```

14.53987730

These results agree. We confidently use the formula to find the distance traveled for a bouncing ball with *h =10* and *r =0.99*. This is a very bouncy ball (very high *r*, almost 1).

```
> s(10, 0.99);
```

1990.

Yes, it sure is a very bouncy ball. The distance traveled for this ball, 1990 meters, is much farther than the balls we've seen in our previous scenarios.

You Try It: Part II

How far does a ball bounce if it is dropped from 10 meters and the ball has *r =.95*?

Determine values for *h* and *r* so the distance traveled is 500 meters. Are these uniques values or are there others that will do this as well?

Part III: Special Function for Distance Traveled

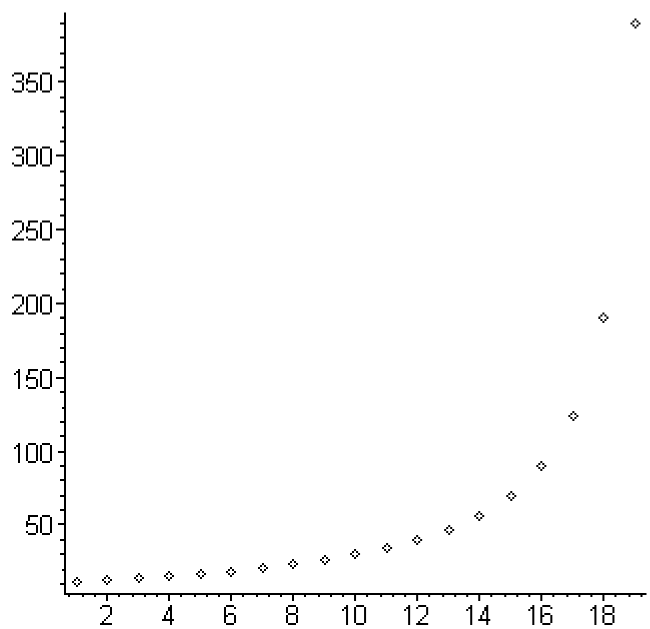
Here we consider the relationship for distance as *r* changes. We'll fix *h* at 10 for these calculations. We consider $u(r) = 10 + 2\left(\sum_{k=1}^{\infty} 10\,(.5e\text{-}1\ n)^k\right)$ for $n=1$ to 19. This produces distances for *r* values from 0.05 to 0.95 in increments of 0.05. First, the function is entered.

```
> u:=n->10+2*sum(10*(0.05*n)^k, k=1..infinity);
```

$$u := n \rightarrow 10 + 2\left(\sum_{k=1}^{\infty} (10\,(0.05\,n)^k)\right)$$

Then, it is plotted for the 19 values of *r*.

```
> data2:=[seq([i,subs(n=i, 10+2*sum(10*(0.05*n)^k, k=1..infinity))], i=1..19)]:
  plots[pointplot](data2);
```



As we should have expected, the values for the distance traveled increase as *n* (and therefore *r*) increases. Did you guess the shape of the curve correctly as well?

You Try It: Part III

Produce a graph of the distances traveled for a ball dropped 25 meters high for 9 different values of *r = .1, .2, .3, .4, .5, .6, .7, .8, and .9.*

Produce a graph of the distances traveled for a ball dropped with *r = .5* for 20 different values of *h* (*h = 10n* , *n = 1* through 20).

Part IV: How Long Does it Take to Bounce an Infinite

Number of Times?

Section 11.2, Exercise 74

It is amazing that the ball bounces an infinite number of times but travels only a finite distance. Let's determine how long it takes to complete these infinite number of bounces. By recalling the formula for the distance traveled for a dropped ball with respect to time *t* (motion under gravity and the hint provided in Exercise 48), we model this motion as $s(t) = 4.9\,t^2$, where *s* is distance in meters and *t* is time in seconds (remember the acceleration due to gravity is $g = 9.8$ *m/sec2*). Solving for *t*, we obtain $t = \sqrt{\frac{s}{4.9}}$. This formula is in effect until the ball strikes the surface. We can easily determine the time to the first bounce if we start at $h = 10$, $t = \sqrt{\frac{10}{4.9}}$.

Knowing the height of each bounce, we can determine the time it takes each bounce (up and back down) to occur. Since the ball has to travel up and down, the distance traveled is twice the bounce height. Therefore, the time to perform the second bounce (up and down) with a coefficient of restitution r is $t = 2\sqrt{\frac{10\,r}{4.9}}$. Similarly, the time for the 5th bounce is $t = 2\sqrt{\frac{10\,r^5}{4.9}}$, and the *kth* bounce is $t = 2\sqrt{\frac{10\,r^k}{4.9}}$. Once again, we can sum these terms to get the amount of time to perform any number of bounces. If we go back to our first scenario in Part I, we have $h = 10$, $r = 0.37$, and number of bounces $n = 10$. Our sum for the total time for these 10 bounces is

$$t = \sqrt{\frac{10}{4.9}} + \left(\sum_{k=1}^{9} 2\sqrt{\frac{10\,.37^k}{4.9}}\right)$$

. Let's compute this.

```
> sqrt(10/4.9)+sum(2*sqrt(10*.37^k/4.9),k = 1 .. 9);
```

5.814620541

So it takes a little over 5.8 seconds to complete 10 bounces. Now we ask the big question. How long does it take to bounce the infinite number of times? We convert this finite sum to an infinite series. Does this series converge? Is it a geometric series? What do you think? Let's compute it, and see what happens. We just change 9 to infinity and simplify the radical to do this computation.

```
> sqrt(10/4.9)*(1+2*sum(.37^(k/2),k = 1 .. infinity));
```

$$5.865198427$$

Yes, the series converges, and the last bounces (those after 10) accumulate very little time. Let's be sure we understand our results using the most general terms we can. If we drop a ball from h meters high and the ball/surface has a coefficient of restitution *r, r < 1*, then the ball bounces an infinite number of times but travels a finite distance in a finite amount of time. We used our knowledge of series (particularly geometric series) to help us understand these principles and to calculate results for specific values of *h* and *r*.

You Try It: Part IV

Find the time to complete 25 bounces of a ball dropped from 50 meters with $r = .8$.

Find the total bouncing time (infinite number of bounces) for a ball dropped from 100 meters with $r = .95$.

Find the total bouncing time (infinite number of bounces) for a ball dropped from 25 feet with $r = .75$. Since the units are in feet, we need to change *g* to $32.2\ feet/sec^2$. The calculation for a 1-foot drop with $r = .8$ is shown.

```
> sqrt(1/16.1)*(1+2*sum(0.8^(k/2), k=1..infinity));
```

$$4.472114380$$

```
> ?

>
```