PRE ALGEBRA

3RD EDITION

HARVEST YOUR MATH SKILLS

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This book is dedicated to everyone who has given us encouragement and support. *Our families, especially our husbands Todd and Gus; our colleagues at Snow College; and our former students, who inspired us to write this book in the first place.*

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PREFACE

In the year 2001, we both began teaching at Snow College in Ephraim, Utah. At this time, the math department assigned us an office to share. Tammy taught classes in the morning and Cindy taught classes in the afternoon. Because of our class times, we did not actually meet until the Spring Semester of 2003. And that is when it all began.

We soon got to know one another and discovered we shared similar teaching methods and expectations from our students. We discussed the frustrations we were experiencing with the current textbook used for the developmental math classes. Supplementing our classes with additional information was something we both had been doing and we decided to combine our efforts. Before long, the extra materials had become the bulk of our curriculum. Our students often commented how they understood the content found in our supplemental materials more than the content from the textbook. After about a year, a student raised his hand and asked, "Why don't we get rid of this book and just use your notes." We decided to do just that, get rid of the book we were using and write our own.

When writing a book, it is helpful to include a Preface, so here is ours. If you are reading this Preface, we want you to know you are among the minority. Most people don't read the Preface of a book. We are grateful you have taken the time to read it, so our time in writing it has not been wasted. Let us now introduce to you the 3rd edition of our textbook.

This book was written out of the desire to make a lasting difference for all of our students, even with all of their differences. Our goal with this book is to help simplify the understanding of math. We have taken a basic approach to Pre Algebra by sticking with one way to work problems, rather than provide multiple ways. We have found that understanding the most simple and general approaches and practicing on multiple examples helps students find success. We have written the book so that it is easy to read and have provided numerous examples for you to follow. We have added and changed a lot from that first go around in 2004. Significant changes in this edition include: new cover photo from Sanpete County, Preface and Dedication, Math Anxiety, How to Be Successful in Your Math Class, Glossary, reorganization of content, addition of content, additional exercise problems, and improved formatting.

We hope as you study Pre Algebra from this book you will enjoy the success that you are looking for. We are grateful to those who have been our inspiration to complete this textbook: our students, colleagues, and families.

If you have any questions, comments, suggestions or concerns regarding this book, feel free to contact us at cindy.alder@snow.edu or tammy.german@snow.edu. Have fun on your journey to learning and making math part of your world!

MATH ANXIETY

Over the course of several years, we have run into different types of students with a variety of levels of math anxiety. Sandra Davis' "Student's Math Anxiety Bill of Rights" and Kathy Acker's "Math Anxiety Code of Responsibilities" have been useful for many students dealing with math anxiety. We hope that you find both of these helpful as you start your math learning experience.

STUDENT'S MATH ANXIETY BILL OF RIGHTS

Adapted from the Math Anxiety Bill of Rights by Sandra Davis, in Donaday & Auslander (1980) *Resource Manual for Counselors/Math Instructor: Math Anxiety, Math Avoidance, Reentry Mathematics*.

1. I have the right to learn at my own pace and not feel put down or stupid if I'm slower than someone else.
2. I have the right to ask whatever questions I have.
3. I have the right to need extra help.
4. I have the right to ask a teacher or tutor for help.
5. I have the right to say I don't understand.
6. I have the right to feel good about myself regardless of my abilities in math.
7. I have the right not to base my self-worth on my math skills.
8. I have the right to view myself as capable of learning math.
9. I have the right to relax.
10. I have the right to be treated as a competent person.
11. I have the right to dislike math.
12. I have the right to define success in my own terms.

MATH ANXIETY CODE OF RESPONSIBILITIES

Adapted from the Math Anxiety Code of Responsibilities by Kathy Acker, retrieved from http://www.mathpower.com/

1. I have the responsibility to attend all classes and do all homework as assigned.
2. I have the responsibility to recognize the rights of others to learn at their own pace.
3. I have the responsibility to seek extra help when necessary.
4. I have the responsibility to see the teacher during office hours or to schedule an appointment for assistance.
5. I have the responsibility to come to class prepared; homework finished and/or questions to ask.
6. I have the responsibility to speak up when I don't understand.
7. I have the responsibility to give math at least the same effort I give to other subjects.
8. I am responsible for my attitudes about my abilities.
9. I have the responsibility for learning and practicing relaxation skills.
10. I have the responsibility to act as a competent adult.
11. I have the responsibility to approach math with an open mind rather than fighting it.
12. I have the responsibility to be realistic about my goals and expectations.

HOW TO BE SUCCESSFUL IN YOUR MATH CLASS

Being successful in a math class, rests on the shoulders of the student. If you do not have a desire to be successful then you have already failed. Here are ten tips for you to follow to make the most out of your math experience.

1. **YOUR ATTITUDE IS EVERYTHING**. Students that come to class knowing that they are never going to get it, usually don't. Students that come to class with the attitude that this is their moment to finally understand math, usually do. If you think and talk like you will succeed most likely you will.
2. **ATTEND AND PARTICIPATE IN CLASS DAILY.** If you are not in class, how are you going to learn? Don't think that you can learn math by sleeping using the book as a pillow, the desk doesn't make a good pillow either. (Don't fall asleep in class!) You need to go to class and actively participate.
3. **MEMORIZE MULTIPLICATION FACTS.** If you don't already have the multiplication facts (0-12) memorized, do it!! There are numerous concepts in math that will be easier to understand when you know your multiplication facts.
4. **READ THE TEXT BOOK.** Read the material in the text book before it is covered in class. Come to class prepared to ask questions when appropriate. Consider math a foreign language – it must be practiced often.
5. **COMPLETE THE ASSIGNED HOMEWORK THE DAY IT IS DISCUSSED IN CLASS.** Don't put off homework until the last minute. If you struggle understanding the homework even after you completed the assigned problems, do additional problems. Come to class with your questions and seek extra help to work through the examples.
6. **KEEP YOUR WORK NEAT AND ORGANIZED.** Allow for enough space to do your work on the paper for both problems in homework and on tests. Number each problem and leave spaces between problems. Do not try to cram it all on one page.
7. **WHEN QUESTIONS ARISE, FIND SOMEONE TO HELP YOU UNDERSTAND THE MATERIAL**. Visit a math lab, schedule an office appointment, work in study groups, work with a tutor, or ask another teacher for help. When working with others, do not let them do the assignment for you. Remember, it is your homework and you are the one that needs to do it. Never leave a question unanswered.
8. **COME INTO OFFICE HOURS WITH QUESTIONS.** Many and most instructors teach because they love students and the process of learning. Office hours are there for additional questions and concerns, take advantage of that time.
9. **TAKE EXAMS AS EARLY AS POSSIBLE.** Begin preparing for tests at least a week in advance. If you are given the opportunity to set your exam time, schedule it as early as possible. Do not put off until the last minute. Review and correct missed problems on your tests. Understand what you missed and why you missed it.
10. **DEVELOP RESPONSIBILITY FOR YOUR OWN SUCCESSES AND FAILURES.** Ultimately, it is up to you whether or not you succeed. You need to be responsible for your actions and the consequences of your actions.

CHAPTER 1

ARITHMETIC WITH SIGNED NUMBERS

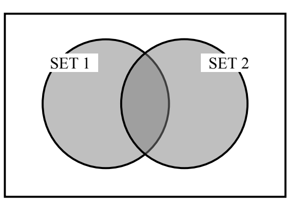
1.1 The Number System and Its Properties

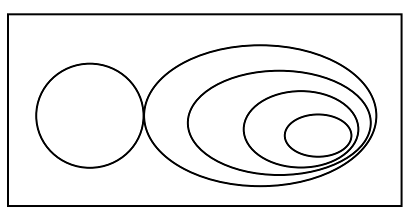
The word or term "set" is used in everyday life. Some sets you may be familiar with are a set of golf clubs, a set of knives, or a set a dishes. A **set** is a collection of objects. The items in a set are known as the **elements** or **members** of that set. If we were to examine the set of letters that form the vowels of the English alphabet, the members in that set would include: *a*, *e*, *i*, *o*, and *u*. Sets can be expressed using what is known as set notation. **Set notation** is a way of writing and using a set of numbers or items. Sets are frequently named using capital letters. The set of vowels in the English alphabet could be named using the capital letter *V*. The elements of the set are then written within braces $\{\ \}$. The set of vowels in the English alphabet would be expressed in set notation as: $V = \{a, e, i, o, u\}$.

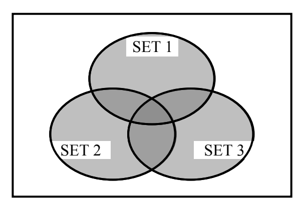
A basic knowledge of the different sets of numbers and their properties is important in order to understand mathematics. These sets of numbers will be examined using set notation as a way to describe the members or elements of the sets. The relationships within these sets of numbers will be visually examined using what is known as a Venn diagram.

The process of visualizing logical relationships was devised by John Venn (1834-1923) a British logician and philosopher. **Venn diagrams,** also referred to as set diagrams, were invented around 1880 and are used as a way of picturing relationships between different groups of things. They are used in many fields, including set theory, probability, logic, statistics, and computer science. A Venn diagram typically consists of a rectangular area representing a universal set. Within the rectangular area there are two or more circular areas which represent groups of items sharing common properties. Any value that belongs to more than one set will be placed in sections where the circles overlap. Some examples of Venn diagrams are pictured below.


John Venn 1834-1923







There are several sets of numbers used in mathematics. Our focus in this section will include the examination of six sets of numbers. These sets include: natural numbers, whole numbers, integers, rational numbers, irrational numbers and real numbers. Set notation and Venn diagrams will be utilized as we study these sets of numbers.

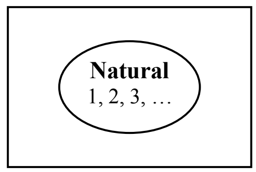
Note: Set notation will not be used to express the sets of rational numbers, irrational numbers, and real numbers because of the complexity of the sets.

NATURAL NUMBERS ($\mathbb{N}$)

When a young child learns to count they begin with 1 and continue on 2, 3, 4, 5... These counting numbers are known as the set of natural numbers. The symbol $\mathbb{N}$ is used to represent the set of natural numbers which include 1, 2, 3, 4, 5, … and is written in set notation as (the use of three dots at the end of the list is a common mathematical notation to indicate that the list keeps going forever):

$$\mathbb{N} = \{1, 2, 3, 4, 5, ...\}$$

The Venn diagram used to illustrate the natural numbers is:

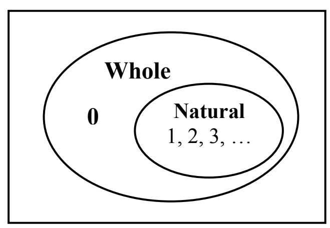


WHOLE NUMBERS ($\mathbb{W}$)

The set of whole numbers is similar to the set of natural numbers. The set of whole numbers includes the set of natural numbers along with zero. The symbol $\mathbb{W}$ is used to represent the set of whole numbers and is written in set notation as:

$$\mathbb{W} = \{0, 1, 2, 3, 4, 5, ...\}$$

The Venn diagram used to illustrate the natural numbers is given below. (Notice the whole numbers circle has the natural numbers circle included within it.)

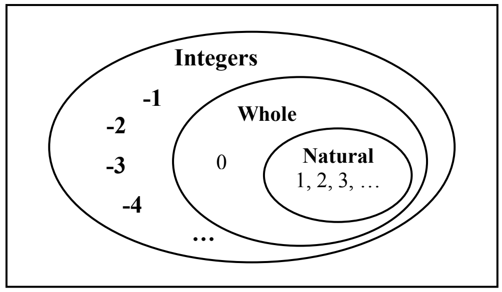


INTEGERS ($\mathbb{Z}$)

The set of integers includes all the whole numbers and their opposites. The symbol $\mathbb{Z}$ (which stands for <u>*Zahlen*</u>, German for *numbers*) is used to represent the set of integers and is written in set notation as:

$$\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$$

The Venn diagram used to illustrate the integers is given below. (Notice the integers circle has the whole numbers and natural numbers circles included within it.)

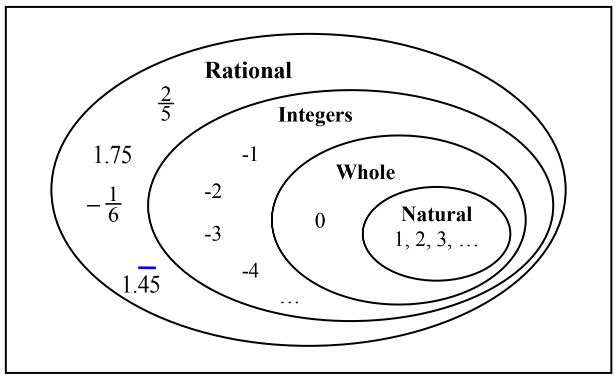


RATIONAL NUMBERS ($\mathbb{Q}$)

The set of rational numbers includes all the integers along with any number that can be expressed as a ratio or fraction $\frac{a}{b}$, where a and b are integers and $b \neq 0$. The decimal form of a rational number is either a terminating or a repeating decimal. The symbol $\mathbb{Q}$ (for quotient, the ratio of two quantities to be divided) is used to represent the set of rational numbers.

Examples of rational numbers include: $\frac{5}{17}$, $\frac{1}{2}$, $1.\overline{62}$ (repeating), -7.34 (terminating), 0, etc.

The Venn diagram used to illustrate the rational numbers is shown below. (Notice the rational circle has the previous three sets of numbers included within it.)

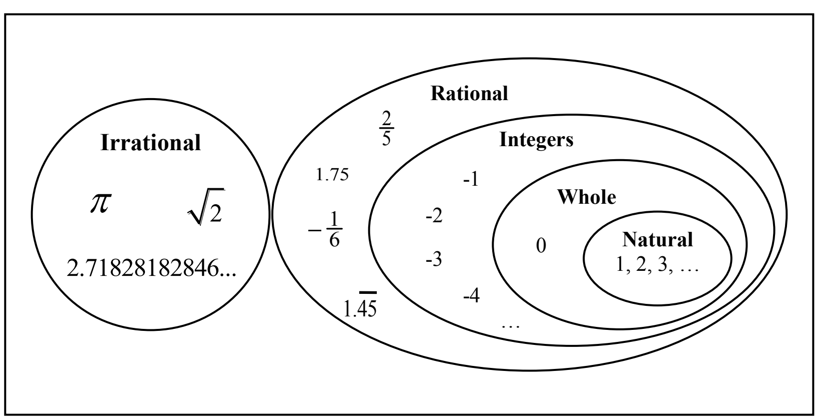


IRRATIONAL NUMBERS ($\mathbb{I}$)

The set of irrational numbers contains any number that is not a rational number. In other words, irrational numbers are numbers that <u>cannot</u> be written as a ratio or fraction. The decimal form of an irrational number is a non-terminating and non-repeating decimal. The symbol $\mathbb{I}$ is used to represent the set of irrational numbers.

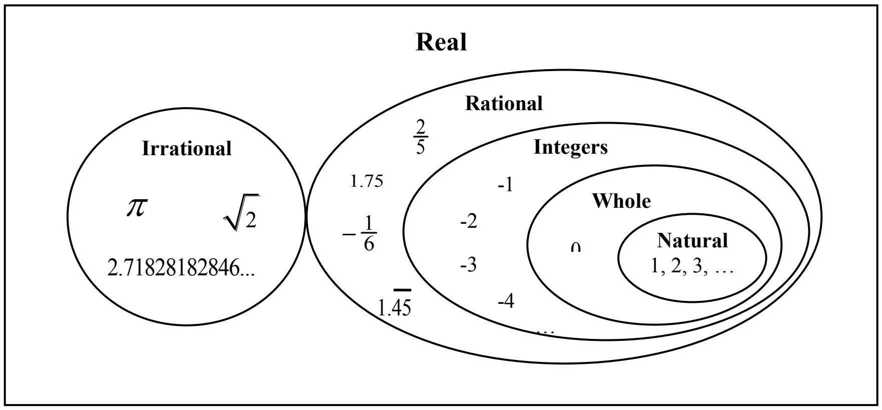
Examples of irrational numbers include: $\pi = 3.14159265...$, $\sqrt{7}$, $-\sqrt{2}$, $e = 2.7182818...$, etc.

The Venn diagram used to illustrate the irrational numbers is shown below. (Notice the irrational circle does not share any elements with the previous four sets of numbers.)

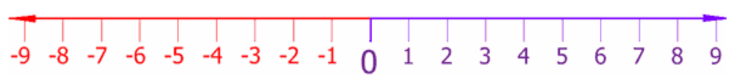


REAL NUMBERS ($\mathbb{R}$)

The set of real numbers includes all the sets of numbers previously discussed. Real numbers can be thought of as the numbers we use in everyday life. The symbol $\mathbb{R}$ is used to represent the set of real numbers. In the following Venn diagram all sets of numbers previously discussed are within the set of real numbers.



The set of real numbers can also be thought of as points on an infinitely long number line. If we were to place a point on the number line, no matter its location, it would be an element of the set of real numbers.



PROPERTIES OF REAL NUMBERS

The numbers we have discussed have actions or operations they can perform. These are referred to as Properties. You may be familiar with many of these properties as you use them frequently in arithmetic problems. It is important for you to learn and use their proper names.

As each property is presented, a definition in mathematical notation will be provided along with two examples that will illustrate the given property. The definitions will use letters or variables to indicate that any real number can be represented in the property.

COMMUTATIVE PROPERTY OF ADDITION

For all real numbers *a* and *b*,

$$a+b=b+a$$

The Commutative Property of Addition states that the order in which two numbers are added does not change the sum. The following examples illustrate this property.

$$2+3=3+2 \qquad\qquad 47+15=15+47$$

$$5=5 \qquad\qquad 62=62$$

COMMUTATIVE PROPERTY OF MULTIPLICATION

For all real numbers *a* and *b,*

$$a \cdot b = b \cdot a$$

The Commutative Property of Multiplication states that the order in which two numbers are multiplied does not change the product. The following examples illustrate this property.

$$4 \cdot 5 = 5 \cdot 4 \qquad 12 \cdot 13 = 13 \cdot 12$$
$$20 = 20 \qquad 156 = 156$$

ASSOCIATIVE PROPERTY OF ADDITION

For all real numbers *a, b,* and *c,*

$$a + (b + c) = (a + b) + c$$

The Associative Property of Addition is similar to the Commutative Property of Addition. The only difference is that the Associative Property holds true for more than two values. It states that the way in which numbers are grouped when added does not change the sum. The following examples illustrate this property.

$$1 + (4 + 7) = (1 + 4) + 7 \qquad 12 + (13 + 51) = (12 + 13) + 51$$
$$1 + 11 = 5 + 7 \qquad 12 + 64 = 25 + 51$$
$$12 = 12 \qquad 76 = 76$$

ASSOCIATIVE PROPERTY OF MULTIPLICATION

For all real numbers *a, b,* and *c,*

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

The Associative Property of Multiplication is similar to the Commutative Property of Multiplication. The only difference is that the Associative Property holds true for more than two values. It states that the way in which numbers are grouped when multiplied does not change the product. The following examples illustrate this property.

$$3 \cdot (2 \cdot 5) = (3 \cdot 2) \cdot 5 \qquad 13 \cdot (10 \cdot 25) = (13 \cdot 10) \cdot 25$$
$$3 \cdot 10 = 6 \cdot 5 \qquad 13 \cdot 250 = 130 \cdot 25$$
$$30 = 30 \qquad 3250 = 3250$$

The next two properties are referred to as identity properties. This name "identity" comes from the fact that when we perform an operation on a number, the number does not change; it holds its identity.

> **IDENTITY PROPERTY OF ADDITION**
>
> For all real numbers *a,*
>
> $$a + 0 = a$$

The Identity Property of Addition states that the sum of any real number and zero is the original real number. The following examples illustrate this property.

$7 + 0 = 7$ $\qquad\qquad$ $1450 + 0 = 1450$

> **IDENTITY PROPERTY OF MULTIPLICATION**
>
> For all real numbers *a,*
>
> $$a \cdot 1 = a$$

The Identity Property of Multiplication states that the product of any real number and one is the original real number. The following examples illustrate this property.

$3 \cdot 1 = 3$ $\qquad\qquad$ $987 \cdot 1 = 987$

There are three special properties that deal with the number zero. The first involves the operation of multiplication while the other two involve the operation of division.

> **MULTIPLICATION PROPERTY OF ZERO**
>
> For all real numbers *a,*
>
> $$a \cdot 0 = 0$$

The Multiplication Property of Zero states that when any real number is multiplied by zero the product will be zero. The following examples illustrate this property.

$19 \cdot 0 = 0$ $\qquad\qquad$ $2547 \cdot 0 = 0$

> **DIVISION INTO ZERO PROPERTY**
>
> For all real numbers *a,* $a \neq 0$,
>
> $$0 \div a = \frac{0}{a} = 0$$

The Division Into Zero Property states that when any non-zero real number is divided into zero the quotient will equal zero. The following examples illustrate this property.

$$0 \div (-27) = 0 \qquad\qquad \frac{0}{13} = 0$$

To understand this property, think of division in terms of multiplication. In our examples, ask the following questions:

"What number times $-27 = 0$?"

"What number times 13 = 0 ?"

The answer to these questions is zero: $-27 \cdot 0 = 0, \quad 13 \cdot 0 = 0$. Therefore, whenever you divide zero by a non-zero number, you will always get zero.

DIVISION BY ZERO PROPERTY

For all non-zero real numbers *a,*

$$a \div 0 = \frac{a}{0} = \text{undefined}$$

The Division By Zero Property states that when any non-zero real number is divided by zero the quotient will be undefined. The following examples illustrate this property.

$$17 \div 0 = \text{undefined} \qquad\qquad \frac{-12}{0} = \text{undefined}$$

To understand this property, again, think of division in terms of multiplication. In our examples, ask the following questions:

"What number times $0 = 17$?"

"What number times $0 = -12$?"

The answer is that you cannot multiply zero by any number to get anything other than zero. Therefore, division by zero cannot be done and is said to be **undefined**.

In the Commutative Properties, the numbers *commute*, or move around. In the Associative Properties, the *association*, or grouping of the numbers changes. In the Identity properties, the *identity* of a number is preserved.

It is important to be familiar with and understand the number system and its properties found in this section. They lay the foundation that you will need in order to comprehend the arithmetic and algebra throughout this book and in future mathematics classes.

1.1 EXERCISES

In 1-12, determine whether the following statements are <u>true</u> or <u>false</u>.

1. All integers are whole numbers.

2. 0 is a natural number.

3. All rational numbers can be written as a ratio or a fraction.

4. $\sqrt{11}$ is a rational number.

5. All natural numbers are rational numbers.

6. All irrational numbers are real numbers.

7. -7 is an integer and a whole number.

8. Real numbers are numbers we use in everyday life.

9. All negative numbers are integers.

10. $\frac{1}{2}$ is a rational number and a real number.

11. All real numbers are rational numbers.

12. All whole numbers are integers.

In 13-18, read and respond to the following exercises.

13. Which set(s) of numbers contains the number 4?

14. What letter is used to represent the set of integers and why?

15. In your own words, define the set of rational numbers.

16. What is the difference between the set of integers and the set of whole numbers? What are their similarities?

17. What letter is used to represent the set of rational numbers and why?

18. State the letter that is used to represent the following sets of numbers. Give one example of a number that belongs to each set.

    a. Natural Numbers
    b. Whole Numbers
    c. Integers
    d. Rational Numbers
    e. Irrational Numbers
    f. Real Numbers

In 19-36, Match each statement with the property that it illustrates. You may need to use a property more than once.

A. Commutative Property of Addition
B. Commutative Property of Multiplication
C. Associative Property of Addition
D. Associative Property of Multiplication
E. Identity Property of Addition
F. Identity Property of Multiplication
G. Multiplicative Property of Zero
H. Division into Zero Property
I. Division by Zero Property

19. $7 \cdot 1 = 1 \cdot 7$

20. $9 \div 0 = \textit{undefined}$

21. $2 + (4 + 5) = (2 + 4) + 5$

22. $9 \cdot 1 = 9$

23. $15 + 0 = 0 + 15$

24. $5 \cdot (2 \cdot 3) = (5 \cdot 2) \cdot 3$

25. $\frac{0}{3} = 0$

26. $(1 + 8) + 3 = 1 + (8 + 3)$

27. $0 = 21 \cdot 0$

28. $4 + 0 = 4$

29. $(4 \cdot 1) \cdot 7 = 4 \cdot (1 \cdot 7)$

30. $3 + 6 = 6 + 3$

31. $10 + (2 + 3) = (10 + 2) + 3$

32. $0 \div 11 = 0$

33. $10 \cdot 2 = 2 \cdot 10$

34. $15 \cdot 0 = 0$

35. $\frac{3}{0} = \textit{undefined}$

36. $3 \cdot 9 = 9 \cdot 3$

In 37-45, read and respond to the following exercises.

37. Describe and give an example of why subtraction is not commutative.

38. Use the commutative property to write an equivalent expression to $5+7$.

39. Give an example of the associative property of addition.

40. In your own words, explain why division by zero is undefined.

41. Using the following example, show why division is not associative.

$$24 \div (4 \div 2) = (24 \div 4) \div 2$$

42. What is the difference between the set of whole numbers and the set of rational numbers?

43. Which sets of numbers are included in the set of real numbers?

44. Which property equals zero, division into zero or division by zero?

45. When would a mathematical statement be undefined?

1.2 Place Value

Every number has a value that is determined by the place it is located. For example, consider money. If you have the number 2 and it is located before the decimal point, 2.00, you have two dollars. If the two is located one space behind the decimal, 0.20, you have twenty cents. If the two is located two spaces behind the decimal, 0.02, you have two cents.

In our number system, each number has a specific place value. Figure 1.1 (below), illustrates this concept.

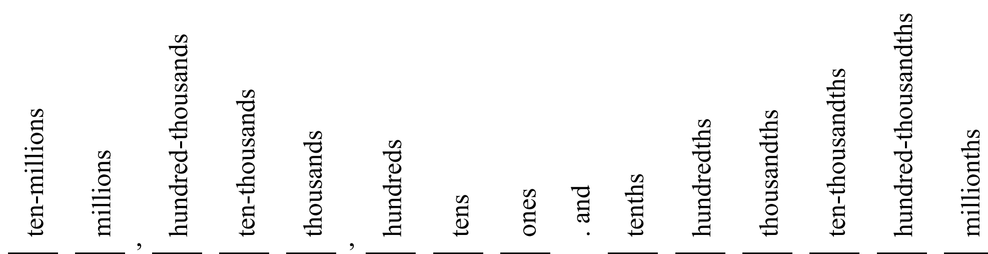


Figure 1.1

It is important to be able to identify the place value a number holds. The numbers located before the decimal point are referred to as integers. (You will notice there is <u>no</u> "**th**" in their name.) The numbers after the decimal point represent a part of an integer, also referred to as a real number. (You will notice there is a **"th"** at the end of their name.)

When reading and writing numbers, **commas** are used in both word form and numerical form. In both forms, the commas are placed after each group of three digits in the numbers located before the decimal point. (Notice where the commas fall in Figure 1.1). Examine the following numbers for the correct location of the commas:

*2**,**180*	*Two thousand**,** one hundred eighty*
*1**,**467**,**210*	*One million**,** four hundred sixty-seven thousand**,** two hundred ten*

Hyphens are used when writing numbers in word form. The numbers between twenty and ninety-nine, that are read as a group of two numbers, need to be hyphenated. For instance:

37	*thirty**-**seven*
149	*one hundred forty**-**nine*

Writing numbers that do not include decimals (integers) in word form is fairly straight forward. Writing numbers that include decimals (real numbers) are more complicated. When writing real numbers in word form, the word "**and**" is used to represent the **decimal point**. The numbers to the left of the decimal point are read as integers. The numbers to the right of the decimal point are read like an integer, but end with the place value name of the last digit.

One last thing to consider when reading and writing numbers, is the sign of the number. If there is no sign in front of your number, it is positive. If the sign ($-$) is in front of your number, it is negative and needs to be read and written as such. For example, the number –1.34 is read *negative one and thirty-four hundredths.*

Example 1 **Write the Integers in words.**

a) **2,179,536**

Two million, one hundred seventy-nine thousand, five hundred thirty-six.

b) **$-$460,254**

Negative four hundred sixty thousand, two hundred fifty-four.

Notice the commas are in the same location in both the numerical form and the written form. Also notice the location of the hyphens.

Example 2 **Write the Integers in digits.**

a) **Negative fifty-four thousand, seven hundred eight.**

$-$54,708

b) **Two hundred three million, fourteen thousand, eighty-five.**

203,014,085

Example 3 **Write the number 23.2 in words.**

Twenty-three and two tenths

Example 4 **Write the number 0.715 in words.**

Seven hundred fifteen thousandths

It is not necessary to say "zero and seven hundred fifteen thousandths." Numbers that only have digits to the right of the decimal do not require the "and" for the decimal point.

Example 5 **Write the number –475,836.02 in words.**

Negative four hundred seventy-five thousand, eight hundred thirty-six and two hundredths

Writing numbers in numerical form is very similar to writing numbers is word form. Start by looking for the decimal place, the "and." Recall that the numbers to the left of the "and" are the integers. The numbers to the right of the "and" are the decimals. The last word in the number will identify the number of decimal places needed. When a place value does not have a digit assigned to it, it is represented with a zero. If there is not an integer part to the number, use a zero for the integer.

For example: **six and seventy-four thousandths**, six is the whole number, a decimal point for the "and", followed by three decimal places (thousandths).

6. ___ ___ ___

Fill in the decimal places with their appropriate numbers.

6. ___ 7 4 — *Use a zero to fill in the unassigned place value.*

6. 0 7 4

6.074

Example 6 **Write seven and four hundredths in digits.**

___ . ___ ___ — *We need one digit to the left of the decimal point because we have a number that only occupies the ones place value. We need two digits to the right of the decimal point for the hundredths.*

7 . ___ ___ — *Fill in the integer, 7.*

7 . 0 4 — *Fill in the decimals, including any 0's for the digits that do not have values assigned to them.*

7.04

Example 7 **Write three thousand, four hundred twenty-seven and four thousand five hundred eighty-nine ten-thousandths in digits.**

___ , ___ ___ ___ . ___ ___ ___ ___ — *We need four digits to the left of the decimal point because we read thousand. We need three digits to the right of the decimal point for the ten-thousandths.*

3 , 4 2 7 . ___ ___ ___ ___ — *Fill in the whole integer, 3427.*

3 , 4 2 7 . 4 5 8 9 — *Fill in the decimals. It is not necessary to include any 0's because all the digits are assigned a number.*

3427.4589

Example 8 **Write negative six thousand three millionths in digits.**

– ___ . ___ ___ ___ ___ ___ ___

Start with a negative sign. We need one digit to the left of the decimal point for the integer and six digits to the right of the decimal point for the millionths.

– 0 . ___ ___ ___ ___ ___ ___

There is no integer stated, so we will represent our integer with a 0.

– 0 . 0 0 6 0 0 3

Fill in the decimals, including any 0's for the digits that do not have values assigned to them.

–0.006003

Application of changing words to digits and digits to words can be seen in the everyday life task of writing checks. When writing a check, the numbers to the left of the decimal place, the dollars, will be written in words, followed by an "**and**" for the decimal. The numbers to the right of the decimal place are the cents. The cents will be left as digits and written over 100. Example 9 properly illustrates how to write a check.

Example 9 **Fill in the amount of the check.**

Mr. Reindeer
North Pole

No. 1225

December 25 20*03*

PAY TO THE
ORDER OF *Santa Claus*

$3,205.75

Three thousand, two hundred five and 75/100 --------------------------DOLLARS

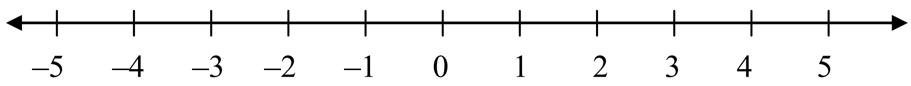
NATIONAL BANK
OF MATHEMATICS
Ephraim, UT 84627

MEMO *Merry Christmas*

Rudolph the Red-Nosed Reindeer

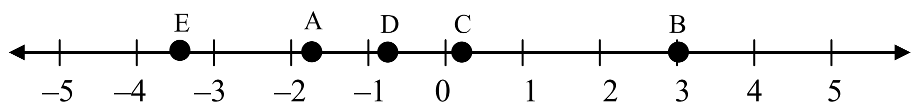
|| 000000|| |:000000000|: 0 000 000 0 ||

While studying place value, it is good to think about where the numbers lie on a number line and how numbers compare one with another. In the figure below, you will find a number line. The numbers to the left are always less than the numbers to the right.



Example 10 **Plot and label the following numbers on the number line below.**

A. –1.6 **B. 3** **C. 0.25** **D. –0.715** **E. –3.4**



When comparing numbers, use the following symbols, $>$, $<$, and $=$. The symbol $>$ means "is greater than," the symbol $<$ means "is less than," and the symbol $=$ means "is equal to." For example:

The statement *2 is less than four* can be written with symbols as: $2 < 4$

Also, the statement *–3 is greater than –7* can be written as: $-3 > -7$

Finally, the statement *5 is equal to 5* can be written as: $5 = 5$

Example 11 **Compare the numbers –2.4 and 2.4. Use $>$, $<$, or $=$.**

$\boxed{2.4 > -2.4}$

2.4 is greater than –2.4, because it is to the right of –2.4 on a number line. It is important to note that a positive number is always larger than a negative number.

Example 12 **Compare the numbers -2.46 and –3.02. Use $>$, $<$, or $=$.**

$\boxed{-3.02 < -2.46}$

–3.02 is less than –2.46 because it is to the left of –2.46 on a number line.

1.2 EXERCISES

In 1-10, write these numbers in words.

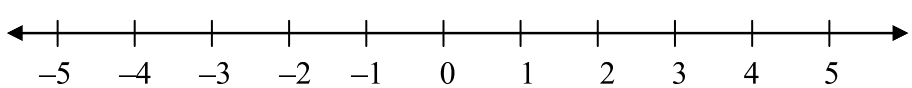
1. 17.1
2. 0.416
3. –34.58
4. 4,785.329
5. –9,472.20073
6. 600.0005
7. -0.0201
8. -190.3709
9. 2,017,010.03
10. -0.003009

In 11-18, change from words into digits.

11. Six and three tenths
12. Negative one hundred twenty-six and seven thousandths
13. Four thousand, two hundred five and six hundredths
14. Three hundred eighty-nine millionths
15. Negative seven million, six hundred thousand, forty-seven and five hundred thirty-nine ten-thousandths
16. One hundred nineteen and twelve hundred-thousandths
17. Two hundred one
18. Three million, six hundred twenty-five thousand, and one tenth

In 19-28, plot and label the following numbers on the number line.

19. (A) 2.5
20. (B) –1.75
21. (C) 4.3
22. (D) –0.2
23. (E) –5
24. (F) 3
25. (G) –4.5
26. (H) 0.5
27. (I) –3.3
28. (J) 0



In 29-40, compare the following numbers. **Use** >, <, or =.

29. –1.347 _____ 1.347

30. 1.72 _____ 0.43

31. –2.68 _____ –3.14

32. 3.29 _____ 3.29

33. 24 _____ 25

34. 8.0024 _____ 8.0025

35. 3.68 _____ –4.53

36. –5.1 _____ –3.1

37. 0.0001 _____ 0.0001

38. –12.35 _____ –12.39

39. –2.11 _____ –21.1

40. 0.0025 _____ 0.025

In 41-45, solve the following application problems.

41. Gus and Todd work for a Construction Company. Gus earns one thousand, six hundred sixteen dollars and twelve cents per month. Todd earns one thousand, five hundred seventy-nine dollars and twenty-five cents per month. Convert their pay into digits. Who makes more money per month?

42. Bradley lost his wallet. In it he had one hundred twenty-seven dollars and thirty-six cents. Scott also lost his wallet with one hundred thirty-six dollars and forty-five cents. Convert the amount each lost into digits. Who lost the least amount of money?

43. Create a number that holds at least seven place values. Write the number in words and in digits.

44. Fill in the amount of the check.

Mr. Easter Bunny | No. 411
Tulip Field, USA | *April 11* 20*04*

PAY TO THE
ORDER OF *Colored Eggs Express* | $12,637.21

______________________________ DOLLARS

NATIONAL BANK
OF MATHEMATICS
Ephraim, UT 84627

MEMO *5 Million Dozens of Eggs* | *Easter Bunny*

|| 000000|| |:000000000|: 0 000 000 0 ||

45. Fill in the amount of the check.

Mrs. Tooth Fairy | No. 000
100 Toothless Avenue
Denture, USA | *June 1* 20*04*

PAY TO THE
ORDER OF *Toothless Grandma* | $7,809.34

______________________________ DOLLARS

NATIONAL BANK
OF MATHEMATICS
Ephraim, UT 84627

MEMO *$$ to help pay for those dentures.* | *Tooth Fairy*

|| 000000|| |:000000000|: 0 000 000 0 ||

1.3 Rounding

In the previous section, the topic of place value was covered. In this section, place value will be used to help with the concept of rounding. We round numbers everyday without knowing we are doing it. For example, if we want to buy an item at the store that has a price tag of $4.99, we say that the item costs $5.00. This is an example of rounding to the nearest dollar.

In the table below, you will find rules to help you when rounding integers.

1. Locate the place value being considered, circle it.
2. Look at and underline the number directly right of the circled number:
   A. If the underlined number is less than five, the circled number stays the same and the digits that follow change to zeros.
   B. If the underlined number is greater than or equal to five, add one to the circled number and the digits that follow will change to zeros.
   (Note: when a 9 is the number in question, and it needs to be increased, the 9 becomes a 0 and the digit to the left of the 9 is increased by one.)

Example 1

Round the following to the nearest <u>hundreds</u>.

a. 12,467 — *4 is the digit in the hundreds place. Circle it.*

12,(4)<u>6</u>7 — *The number to the right is a 6, underline it. Because 6 is greater than 5, increase the 4 by one and replace the digits that follow with zeros.*

12,500

b. 14 — *There is not a digit in the hundreds place. When this occurs, we consider that number to be a 0.*

(0)<u>1</u>4 — *The number to the right of the place value is a 1, underline it. Because 1 is less than 5, leave the 0 and replace the digits that follow with 0's.*

0 0 0

0

Example 2 **Round the following to the nearest <u>thousands</u>.**

a. 157,243

7 is the digit in the thousands place. Circle it.

15(7),<u>2</u>43

The number to the right is a 2, underline it. Because 2 is less than 5, leave the 7 and replace the digits that follow with 0's.

157,000

b. 19,860

9 is the digit in the thousands place. Circle it.

1(9),<u>8</u>60

The number to the right is an 8 underline it. Because 8 is greater than 5, increase the 9 by one and replace the digits that follow with 0's. Recall, when a 9 is increased it becomes a 10. This will require the number to the left of the 9 to be increased by one and the 9 will change to a 0.

20,000

Rounding decimal numbers is quite similar to rounding integers. There is really only one difference. Any digits after the place value being considered will be dropped, if they occur <u>after</u> the decimal point. For example: round the number 127.6475 to the nearest hundredths.

127.6475

4 is the digit in the hundredths place. Circle it.

127.6(4)<u>7</u>5

The number to the right is a 7, underline it. Because 7 is greater than 5, increase the 4 by one and drop all the digits after the decimal point.

127.65

When rounding decimals, the final answer must have exactly the number of digits named. For example, if you are asked to round a problem to the nearest thousandths, you must have three decimal places in your answer. This may require the use of zeros as in examples 3b and 4a.

Example 3 **Round the following to the nearest <u>tenths</u>.**

a. 16,047.346

3 is the digit in the tenths place. Circle it.

16,047.(3)<u>4</u>6

The number to the right is a 4, underline it. Because 4 is less than 5, leave the 3 and drop all the digits after the tenths place.

16,047.3

b. 547.96

9 is the digit in the tenths place. Circle it.

547.(9)6

The number to the right is a 6, underline it. Because 6 is greater than 5, increase the 9 by one and drop all the digits after the tenths place. ***Remember the rule of rounding with 9, when the 9 is increased it becomes a 10. This will require the number to the left of the 9 to be increased by one and the 9 will change to a 0. Keep the last 0 so that the answer ends in the tenths place.***

548.0

Example 4

Round the following to the nearest hundredths.

a. 3,426.795

9 is the digit in the hundredths place. Circle it.

3,426.7(9)5

The number to the right is a 5, underline it. Because 5 is equal to 5, increase the 9 by one and drop all the digits after the hundredths place. ***Recall, when working with 9, and it needs to be increased, the change also involves the digit to the left of the 9. The 9 becomes a 0 and the digit to the left of the 9, is increased buy one. Keep the last 0 so that the answer ends in hundredths.***

3,426.80

b. 10.3428

4 is the digit in the tenths place. Circle it.

10.3(4)28

The number to the right is a 2, underline it. Because 2 is less than 5, leave the 4 and drop all the digits after the hundredths place.

10.34

Example 5

Round the following to the nearest ones.

a. 0.12

0 is the digit in the one's place. Circle it.

(0).12

The number to the right is a 1, underline it. Because 1 is less than 5, leave the 0 and drop all the digits after the decimal point.

0

b. 15.63

5 is the digit in the one's place. Circle it.

1(5).63

The number to the right is a 6, underline it. Because 6 is greater than 5, increase the 5 by one and drop all the digits after the decimal point.

16

Example 6

Round the following to the nearest cent.

a. $83.925

$83.9(2)5

2 is the digit in the cent's place. (Cents is located in the hundredths place.) Circle it.

$83.93

The number to the right is a 5, underline it. Because 5 is equal to 5, increase the 2 by one and drop all the digits after the cent's place.

b. $47.884

8 is the digit in the cent's place. (Cents islocated in the hundredths place.) Circle it.

$47.8(8)4

The number to the right is a 4, underline it. Because 4 is less than 5, leave the 8 and drop all the digits after the cent's place.

$47.88

Example 7

Round 1,745.9748 to the indicated place value.

a. **Thousands**

(1),745.9748

1 is the digit in the thousand's place. Circle it.

2,000

The number to the right is a 7, underline it. Because 7 is greater than 5, increase the 1 by one and replace the digits that follow with 0's and drop the digits after the decimal point.

b. **Hundreds**

1,(7)45.9748

7 is the digit in the hundred's place. Circle it.

1,700

The number to the right is a 4, underline it. Because 4 is less than 5, leave the 7 and replace the digits that follow with 0's and drop the digits after the decimal point.

c. **Tens**

1,7(4)5.9748

4 is the digit in the ten's place. Circle it.

1,750

The number to the right is a 5, underline it. Because 5 is equal to 5, increase the 4 by one and replace the digits that follow with 0's and drop the digits after the decimal point.

d. **Ones**

1,74(5).9748

5 is the digit in the one's place. Circle it.

1,746

The number to the right is a 9, underline it. Because 9 is greater than 5, increase the 5 by one and replace the digits that follow with 0's and drop the digits after the decimal point.

e. **Tenths**

1,745.9748 — *9 is the digit in the tenth's place. Circle it.*

1,746.0 — *The number to the right is a 7, underline it. Because 7 is greater than 5, increase the 9 by one and drop all the digits after the tenth place.* ***Recall, when a 9 is the number in question, and it needs to be increased, the change also involves the digit to the left of the 9. Keep the last 0 so that the answer ends in tenths.***

f. **Hundredths**

1,745.9748 — *7 is the digit in the hundredth's place. Circle it.*

1,745.97 — *The number to the right is a 4, underline it. Because 4 is less than 5, leave the 7 and drop all the digits after the hundredths place.*

g. **Thousandths**

1,745.9748 — *4 is the digit in the thousandth's place. Circle it.*

1,745.975 — *The number to the right is an 8, underline it. Because 8 is greater than 5, increase the 4 by one and drop all the digits after the thousandths place.*

Example 8 **Complete the following table by rounding to the indicated place value.**

Number	Thousands	Hundreds	Tens	Ones	Tenths	Hundredths	Thousandths
855.4932	1,000	900	860	855	855.5	855.49	855.493
126.79	0	100	130	127	126.8	126.79	126.790
9.485732	0	0	10	9	9.5	9.49	9.486
0.127	0	0	0	0	0.1	0.13	0.127
0.0024	0	0	0	0	0.0	0.00	0.002
2,015	2,000	2,000	2,020	2,015	2,015.0	2,015.00	2,015.000
4,548.065	5,000	4,500	4,550	4,548	4,548.1	4,548.07	4,548.065

Application of rounding can be seen in our everyday activities. For instance, when dealing with money, values are frequently rounded to the nearest dollar. When rounding problems that involve money, two decimal places are left in the answer regardless the place value we round to. This is done to represent both dollars and cents in our answer. The two decimal places will be zero's when rounding to the nearest dollar. When rounding to the nearest cent, these two decimal places will often be numbers other than zero. The following example illustrates this idea.

Example 9

At the school bookstore you need to purchase a calculator for your math course. It costs $12.75. To the nearest dollar, how much money do you need in order to make the purchase?

To determine the number of dollars needed, we must round $12.75 to the nearest dollar.

$12.75	*2 is the digit in the dollar's place. Circle it.*
$1(2).<u>7</u>5	*The number to the right is a 7, underline it. Because 7 is greater than 5, increase the 2 by one. Because we are dealing with money we will leave two decimal places in our answer. These two decimal places will become zeros.*
$13.00	

*Therefore, you need **<u>$13.00</u>** to make the purchase.*

The rules that have been discussed are general rules for rounding. Some situations do not follow these general rules. Closely examine the following example.

Example 10

It is Trevor's birthday and his roommates are throwing him a surprise party. Unfortunately, they forgot to purchase paper plates for the cake. Since they hate doing dishes, they decide to run to the local store to buy some paper plates. Trevor is very popular and his roommates expect about fifty-three people at the party. When they arrive to the store they find that the paper plates are sold in packages of ten. How many packages will they need to buy?

Since paper plates are sold in packages of 10, we must round 53 to the nearest tens. In our rounding rules, 53 rounds to 50.

The paper plates are sold in packages of 10. $50 \div 10 = 5$. *5 packages will be needed.*

Are 5 packages, 50 plates, enough? No. We will be short 3 plates. Therefore, Trevor's roommates must purchase an additional package giving them a total of ***6 packages, 60 plates.***

Example 10 is one application problem where the general rules of rounding do not necessarily apply. Carefully read your application problems to determine how to round.

1.3 EXERCISES

In 1-26 round to the indicated place value.

1. 13 *nearest tens*
2. 3 *nearest tens*
3. 48 *nearest tens*
4. 412 *nearest hundreds*
5. 987 *nearest hundreds*
6. 53 *nearest hundreds*
7. 2.47 *nearest tenths*
8. 123.2354 *nearest tenths*
9. 12.063 *nearest tenths*
10. 0.2745 *nearest hundredths*
11. 45.09864 *nearest hundredths*
12. 347.0005 *nearest thousandths*
13. 1,434.0572 *nearest thousandths*
14. 0.2476543 *nearest thousandths*
15. 0.67 *nearest ones*
16. 23.27 *nearest ones*
17. 243.504 *nearest ones*
18. $45.687 *nearest cent*
19. $2.0341 *nearest cent*
20. $4,305.706 *nearest cent*
21. $654.05 *nearest dollar*
22. $7,033.75 *nearest dollar*
23. $400.50 *nearest dollar*
24. $789.25 *nearest hundred dollar*
25. $12,532.00 *nearest hundred dollar*
26. $13,025.12 *nearest hundred dollar*

In 27-34, complete the following table by rounding to the indicated place value.

	Number	Thousands	Hundreds	Tens	Ones	Tenths	Hundredths	Thousandths
27.	975.6342							
28.	12.345							
29.	5.68973							
30.	23,482.13							
31.	1005							
32.	98.0959							
33.	0.0348							
34.	5,648.053							

In 35-39, solve the following application problems.

35. Beth and Tamee are planning a shopping trip to a large city over the Thanksgiving break, a distance of 543 miles. Estimate the distance to the nearest hundred miles.

36. Amie is doing her student teaching this semester. She wants to take treats to her school for twenty-three students. She goes to the grocery store and decides to buy candy bars. They are sold in packages of five. How many packages will she need to buy to have enough candy bars for her class? How many candy bars will be left over?

37. Cody goes to the local gravel pit. He knows that he needs to cover 439 square yards of ground with the gravel. One dump truck load will cover about 50 square yards. How many loads of gravel will he need?

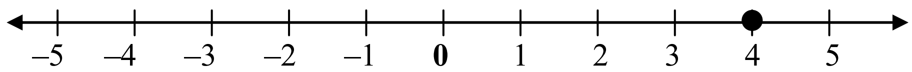
38. Danny needs to get his girlfriend a Valentines gift. He picked out a card that cost $2.97, a box of chocolates that cost $4.98, and a stuffed animal that costs $6.99. How many five dollar-bills will he need to cover the cost of his Valentines gift?

39. Late in 1995, the national debt was reported to be approximately $4,988,882,588,134. How would you write that number in words? How might you round that number so it would be easier to discuss and still convey the size of the number?

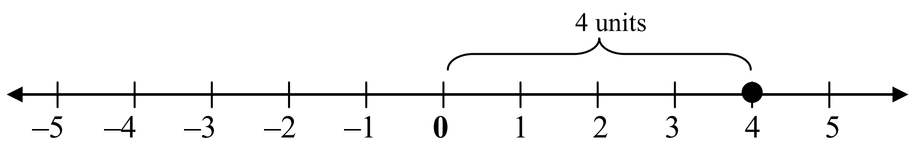
1.4 Absolute Value & Opposites

ABSOLUTE VALUE

The **absolute value** of a number is the distance that the given number is from zero on a number line. The notation for absolute value is $|\ \ |$. For example, $|4|$ is read as "the absolute value of four." To find $|4|$, we will first draw a number line and plot the number 4.

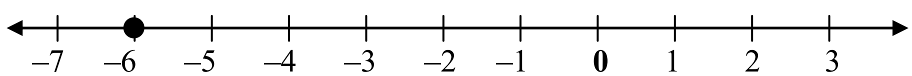


Now, find the distance from zero to 4.

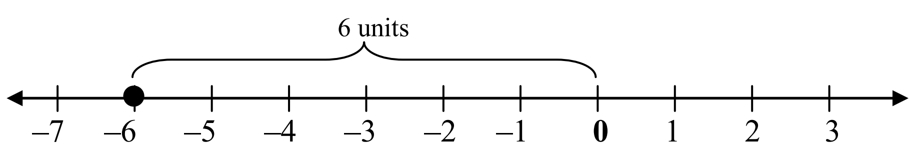


Therefore, $|4| = \boxed{4}$.

Using the same method as above, we will find the value of $|-6|$. First draw a number line and plot the number –6.



Now, find the distance from zero to –6.



Therefore, $|-6| = \boxed{6}$.

Example 1

Find the value of $|5|$.

$|5| = \boxed{5}$

5 is 5 units from zero on the number line, so its absolute value is equal to 5.

Example 2

Find the value of $|-3|$.

$|-3| = \boxed{3}$

–3 is 3 units from zero on the number line, so its absolute value is equal to 3.

Example 3 **Find the value of $|-4.5|$.**

$|-4.5| = \boxed{4.5}$

–4.5 is 4.5 units from zero on the number line, so its absolute value is equal to 4.5.

Example 4 **Find the value of $|0|$.**

$|0| = \boxed{0}$

0 is 0 units from zero on the number line, so its absolute value is equal to 0.

Note: The absolute value of a number is the numbers distance from zero on a number line. Distance can never be a negative value. Therefore, the absolute value of a number is always equal to 0 or a positive number. An absolute value is never negative.

What is the difference between $|-6|$ and (-6)? We know that $|-6| = 6$ because the number –6 is 6 units from zero on the number line. When a number is enclosed in parentheses, ( ), it does <u>not</u> mean absolute value or distance, it simply means the number holds its value. So, $(-6) = -6$

Example 5 **Find the value of (-5).**

$(-5) = \boxed{-5}$

Parentheses mean that the number holds its value.

Example 6 **Find the value of (10).**

$(10) = \boxed{10}$

Parentheses mean that the number holds its value.

Example 7 **Find the value of (-3.6).**

$(-3.6) = \boxed{-3.6}$

Parentheses mean that the number holds its value.

OPPOSITE

Two real numbers are **opposites** if they are represented on the number line by points that are the same distance from zero, but in opposite directions from zero. For example the opposite of 5 is –5. The opposite of a number is also known as the additive inverse. The following property illustrates the relationship between a number and its additive inverse or opposite.

ADDITIVE INVERSE PROPERTY

For all real numbers *a*,

$$a+(-a)=0$$

The Additive Inverse Property states that the sum of any number and its additive inverse is zero. The additive inverse is also known as the opposite of a number. For example:
$3+(-3)=0$ and $-2+(2)=0$

The additive inverse or opposite of any positive number is a negative number and the additive inverse or opposite of any negative number is a positive number. Zero is its own opposite. To show the opposite we use a negative sign, –.

Example 8

Find the value of $-|15|$.

$-|15|=\boxed{-15}$

We are asked to find the opposite of the absolute value of 15. The absolute value of 15 is 15 and the opposite of 15 is –15.

Example 9

Find the value of $-(2)$.

$-(2)=\boxed{-2}$

We are asked to find the opposite of 2, which is –2.

Example 10

Find the value of $-\left|\frac{1}{4}\right|$.

$-\left|\frac{1}{4}\right|=\boxed{-\frac{1}{4}}$

We are asked to find the opposite of the absolute value of $\frac{1}{4}$. The absolute value of $\frac{1}{4}$ is $\frac{1}{4}$ and the opposite of $\frac{1}{4}$ is $-\frac{1}{4}$.

Example 11

Find the value of $-|-11|$.

$-|-11|=\boxed{-11}$

We are asked to find the opposite of the absolute value of –11. The absolute value of –11 is 11 and the opposite of 11 is –11.

Example 12 **Find the value of $-(-7.5)$.**

$-(-7.5) = \boxed{7.5}$

We are asked to find the opposite of –7.5, which is 7.5.

The concepts of simplifying absolute value problems and finding the opposite of a number have been presented. These concepts will now be used in comparison problems. The following symbols are used when comparing numbers: $>$, $<$, and $=$. Recall, the symbol $>$ means "is greater than," the symbol $<$ means "is less than," and the symbol = means "is equal to."

Example 13 **Compare the following numbers. Insert $>$, $<$, or $=$ between the pair of numbers.**

(-9) ____ $-|11|$

Simplify each number.

$-9 \underline{\boxed{>}} -11$

–9 is greater than –11.

Example 14 **Compare the following numbers. Insert $>$, $<$, or $=$ between the pair of numbers.**

$|-4|$ ____ $|4|$

Simplify each number.

$4 \underline{\boxed{=}} 4$

4 is equal to 4.

Example 15 **Compare the following numbers. Insert $>$, $<$, or $=$ between the pair of numbers.**

$-(21)$ ____ (-17)

Simplify each number.

$-21 \underline{\boxed{<}} -17$

–21 is less than –17.

Example 16 **Compare the following numbers. Insert $>$, $<$, or $=$ between the pair of numbers.**

$-|-5.4|$ ____ $-(-5.4)$

Simplify each number.

$-5.4 \underline{\boxed{<}} 5.4$

–5.4 is less than 5.4.

1.4 EXERCISES

In 1-16, simplify.

1. $|2|$
2. $|-10|$
3. (-5)
4. $-|-19|$
5. $|-0.3|$
6. $\left|\frac{1}{2}\right|$
7. $-(-7)$
8. $|0|$
9. $-|4|$
10. $|3.7|$
11. $-(8.3)$
12. $-|-7.5|$
13. $|8|$
14. $|1,247|$
15. (5.7)
16. $|-4,278|$

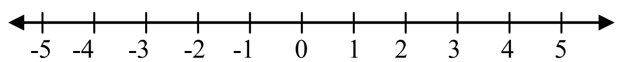
In 17-28, Compare the following numbers. Insert $>$, $<$, or $=$ ***between the pair of numbers.***

17. -3 ____ -5
18. $|-33|$ ____ $-(-33)$
19. $-|17|$ ____ $-(-17)$
20. $|0|$ ____ $|-9|$
21. -18 ____ -6
22. $|-9|$ ____ $|-14|$
23. $|-8|$ ____ $|-4|$
24. -45 ____ 0
25. $-|-2|$ ____ $-|-10|$
26. $-|-12|$ ____ $-(-18)$
27. $-|-8|$ ____ $-|-4|$
28. $-(-7)$ ____ $|-7|$

1.5 Addition & Subtraction of Integers

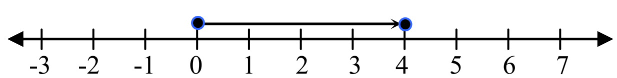
ADDITION

Addition is the first of the four basic arithmetic operations and is the process of combining quantities. Understanding addition of signed numbers, or integers, is a fundamental math skill. One way to gain this math skill is with the use of a number line. When adding integers, start by drawing a number line.



The numbers to the right of zero are always positive. The numbers to the left of zero are always negative. To add using a number line, start at the origin (zero). Draw an arrow to represent your first value. If the value is positive, use an arrow pointing to the right in the positive direction. If the value is negative, use an arrow pointing to the left in the negative direction.

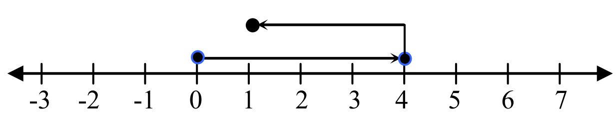
For example: $4+(-3)$



Start at zero and use an arrow pointing to the right with a length of 4.

Next, draw an arrow representing the second value. Start at the tip of the first arrow, making sure to use an arrow pointing in the proper direction depending on the sign of the second value.

In the example, $4+(-3)$, start at the tip of the first arrow and draw a second arrow which will point in the negative direction, left. It will have a length of three units.

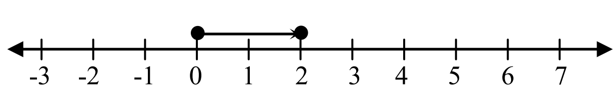


Draw the second arrow.

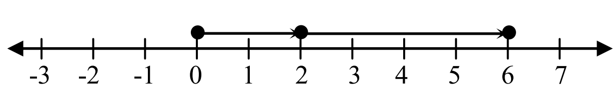
The point of the second arrow ends at the sum. Therefore,

$$4+(-3)=\boxed{1}$$

Example 1 **Find the sum of 2 + 4 .**



Start at zero. Since the 2 is positive, draw an arrow 2 units long pointing in the positive direction (right).

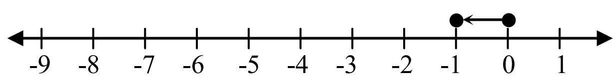


Beginning at the tip of the first arrow, draw a second arrow, 4 units long pointing in the positive direction (right).

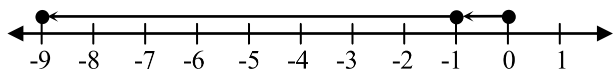
$$2+4=\boxed{6}$$

The point we end on is our answer, 6.

Example 2 **Find the sum of** $(-1)+(-8)$.



Start at zero. Since the 1 is negative, draw an arrow 1 unit long pointing in the negative direction (left).

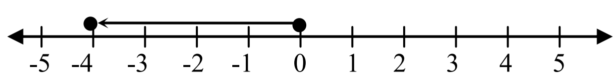


Beginning at the tip of the first arrow, draw a second arrow, 8 units long pointing in the negative direction (left).

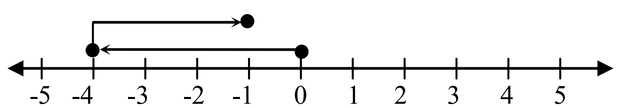
$$(-1)+(-8)=\boxed{-9}$$

The point we end on is our answer, -9.

Example 3 **Find the sum of** $(-4)+3$.



Start at zero. Since the 4 is negative, draw an arrow 4 units long pointing in the negative direction (left).

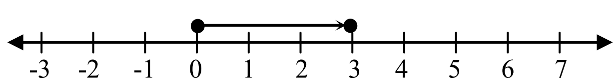


Beginning at the tip of the first arrow, draw a second arrow, 3 units long pointing in the positive direction (right).

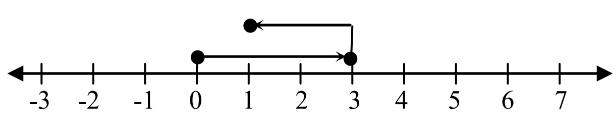
$$(-4)+3=\boxed{-1}$$

The point we end on is our answer, -1.

Example 4 **Find the sum of** $3+(-2)$.



Start at zero. Since the 3 is positive, draw an arrow 3 units long pointing in the positive direction (right).



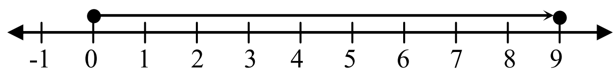
Beginning at the tip of the first arrow, draw a second arrow, 2 units long pointing in the negative direction (left).

$$3+(-2)=\boxed{1}$$

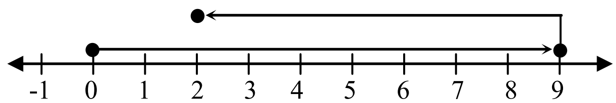
The point we end on is our answer,1.

Did you notice that adding a negative, it is the same as subtracting a positive? The problem in example 4 can also be written as $3-2$. The answer will remain 1.

Example 5 **Find the sum of $9+(-7)$.**



Start at zero. Since the 9 is positive, draw an arrow 9 units long pointing in the positive direction (right).



Beginning at the tip of the first arrow, draw a second arrow, 7 units long pointing in the negative direction (left).

$$9+(-7)=\boxed{2}$$

The point we end on is our answer, 2.

When adding and subtracting integers with multiple digits, it is not be practical to make use of a number line. To add these types of numbers, stack the numbers vertically lining up the place value of the digits. Start by adding the numbers in the ones column and then move to add the numbers in the tens column, hundreds column, etc. If the answer in one column is greater than nine, it will be necessary to carry. The following example demonstrates carrying when adding integers.

Example 6 **Find the sum of $39+17$.**

$$\begin{array}{r} {}^{1} \\ 39 \\ +17 \\ \hline 56 \end{array}$$

The integers are both positive, stack the numbers and add each column.

Add the numbers in the ones place value column. The sum of 9 and 7 is 16. Place the 6 below the line in the ones place value column. Carry the 1 to the tens place value column. Add the numbers in the tens place value column.

$$\boxed{56}$$

The sum is positive because the original integers were positive.

The following table describes how to set up a problem involving addition of integers.

ADDITION OF INTEGERS
1. To add integers that have like signs, add the absolute value of the integers. The sum has the same sign as the original integers.
2. To add integers that have unlike signs, subtract the smaller absolute value from the larger absolute value. The sum has the sign of the integer with the greater absolute value.

Example 7

Find the sum of $(-36)+145$ **.**

$$\begin{array}{r} 1\overset{3}{\not{4}}\overset{1}{5} \\ -\;3\;6 \\ \hline 1\;0\;9 \end{array}$$

The integers have different signs so subtract the smaller absolute value from the larger absolute value.

The sign on the answer will be positive because the integer with the greatest absolute value is positive.

$\boxed{109}$

Example 8

Find the sum of $(-921)+(-44)$ **.**

$$\begin{array}{r} 921 \\ +44 \\ \hline 965 \end{array}$$

The integers have the same sign (both negative) so add the absolute value of the integers.

The sum will be negative because the original integers were negative.

$\boxed{-965}$

Example 9

Find the sum of $(-479)+251$ **.**

$$\begin{array}{r} 4\;7\;9 \\ -\;2\;5\;1 \\ \hline 2\;2\;8 \end{array}$$

The integers have different signs so subtract the smaller absolute value from the larger absolute value.

The sign on the answer will be negative because the integer with the greatest absolute value is negative.

$\boxed{-228}$

SUBTRACTION

Subtraction is the second of the four basic arithmetic operations and is the process of removing one quantity from another quantity. Subtraction is the inverse of addition. This means that if we start with any number then add any number to it, we can turn around and subtract the number that was added returning to the original number. Because addition and subtraction are inverse operations, a subtraction problem can be rewritten as an addition problem. This concept will help when subtracting integers. To subtract two integers, take the first integer and add the opposite of the second integer. Examine the following examples.

subtraction problem	=	*first integer*	+	*opposite of second integer*		
$7-5$	=	7	+	(-5)	=	$\boxed{2}$
$-3-8$	=	-3	+	(-8)	=	$\boxed{-11}$
$4-(-1)$	=	4	+	1	=	$\boxed{5}$
$-12-(-6)$	=	-12	+	6	=	$\boxed{-6}$

SUBTRACTION OF INTEGERS

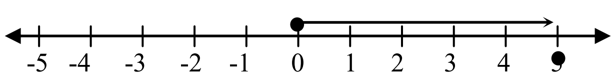
To subtract integers, rewrite your subtraction problem as an addition problem.

subtraction problem = *first number* + *opposite of second number*

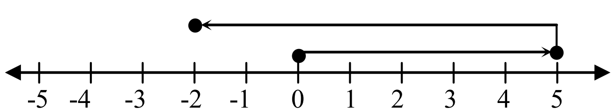
Example 10 **Find the difference of** $5-7$.

Rewrite as an addition problem.

$$5+(-7)$$



Start at zero. Since the 5 is positive, draw an arrow 5 units long pointing in the positive direction (right).



Beginning at the tip of the first arrow, draw a second arrow, 7 units long pointing in the negative direction (left).

$$5-7=\boxed{-2}$$

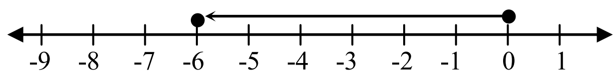
The point we end on is our answer, -2.

Example 11

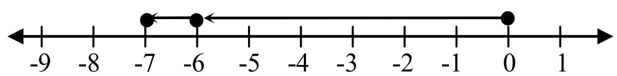
Find the difference of $-6-1$.

Rewrite as an addition problem.

$$-6+(-1)$$



Start at zero. Since the 6 is negative, draw an arrow 6 units long pointing in the negative direction (left).



Beginning at the tip of the first arrow, draw a second arrow, 1 unit long pointing in the negative direction (left).

$$-6-1=\boxed{-7}$$

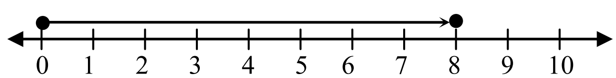
The point we end on is our answer, -7.

Example 12

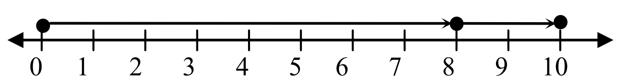
Find the difference of $8-(-2)$.

Rewrite as an addition problem.

$$8+2$$



Start at zero. Since the 8 is positive, draw an arrow 8 units long pointing in the positive direction (right).



Beginning at the tip of the first arrow, draw a second arrow, 2 units long pointing in the positive direction (right).

$$8-(-2)=\boxed{10}$$

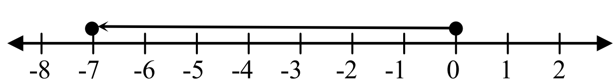
The point we end on is our answer, 10.

Example 13

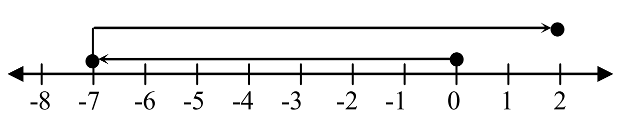
Find the difference of $(-7)-(-9)$.

Rewrite as an addition problem.

$$(-7)+9$$



Start at zero. Since the 7 is negative, draw an arrow 7 units long pointing in the negative direction (left).



Beginning at the tip of the first arrow, draw a second arrow, 9 units long pointing in the positive direction (right).

$$(-7)-(-9)=\boxed{2}$$

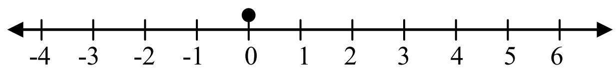
The point we end on is our answer, 2.

Example 14

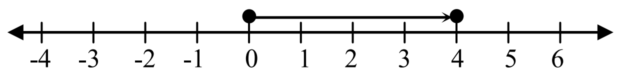
Find the difference of $0-(-4)$.

Rewrite as an addition problem.

$$0+4$$



Start at zero. Since the first number is zero, there is no need to draw an arrow.



Beginning at zero, draw an arrow, 4 units long pointing in the positive direction (right).

$$0-(-4)=\boxed{4}$$

The point we end on is our answer, 4.

When subtracting larger integers, again it is not necessarily practical to draw a number line. Change the subtraction problem into an addition problem and follow the steps for addition of integers.

Example 15

Find the difference of $451-380$.

$$451-380$$

Rewrite as an addition problem.

$$451+(-380)$$

The integers have different signs so subtract the smaller absolute value from the larger absolute value.

$$\begin{array}{r} \overset{3}{\not{4}}\,\overset{1}{5}\,1 \\ -\,3\,8\,0 \\ \hline 7\,1 \end{array}$$

$$\boxed{71}$$

The sign on the answer will be positive because the integer with the greatest absolute value is positive.

Example 16

Find the difference of $(-480)-720$.

$$(-480)-720$$

Rewrite as an addition problem.

$$(-480)+(-720)$$

$$\begin{array}{r} \overset{1}{4}80 \\ +720 \\ \hline 1,200 \end{array}$$

The integers have the same sign (both negative) so add the absolute value of the integers.

The sum will be negative because the original integers were negative.

$$\boxed{-1,200}$$

Example 17 **Find the difference of** $(-365)-(-2{,}144)$.

$(-365)-(-2{,}144)$

$(-365)+2{,}144$

Rewrite as an addition problem.

$$\begin{array}{r} \overset{1}{\not2},\overset{^{1}0}{\not1}\,\overset{^{1}3}{\not4}\,\overset{1}{4} \\ -3\;6\;5 \\ \hline 1{,}7\;7\;9 \end{array}$$

The integers have different signs so subtract the smaller absolute value from the larger absolute value.

$\boxed{1{,}779}$

The sign on the answer will be positive because the integer with the greatest absolute value is positive.

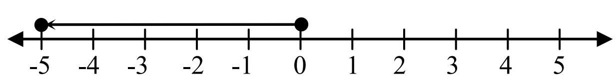
The following examples illustrate problems that involve addition and subtraction of more than two numbers.

Example 18 **Simplify.** $(-5)-0+10$

Rewrite as an addition problem.

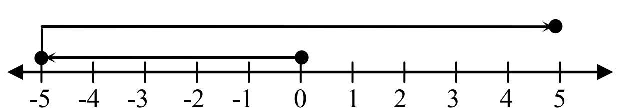
$(-5)+0+10$

(Because zero is neither positive nor negative, the sign in front of the zero does not matter and can be changed to a positive.)



Start at zero. Since the 5 is negative, draw an arrow 5 units long pointing in the negative direction (left).

The second value is 0. There is no need to draw an arrow for this value.



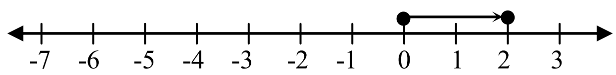
Beginning at the tip of the first arrow, draw a second arrow, 10 units long pointing in the positive direction (right).

$-5-0+10=\boxed{5}$

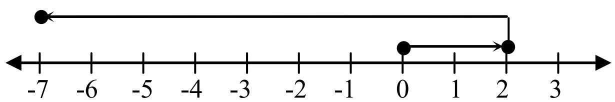
The point we end on is our answer, 5.

Example 19 **Simplify.** $2-9+8-4$ *Rewrite as an addition problem.*

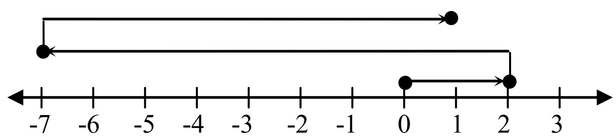
$$2+(-9)+8+(-4)$$



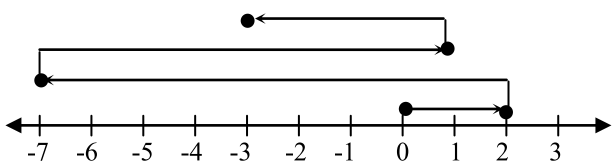
Start at zero. Since the 2 is positive, draw an arrow 2 units long pointing in the positive direction (right).



Beginning at the tip of the first arrow, draw a second arrow, 9 units long pointing in the negative direction (left).



Beginning at the tip of the second arrow, draw a third arrow, 8 units long pointing in the positive direction (right).



Beginning at the tip of the third arrow, draw a fourth arrow, 4 units long pointing in the negative direction (left).

$$2-9+8-4=\boxed{-3}$$

The point we end on is our answer, -3.

CHANGING WORDS TO SYMBOLS

In mathematics and in life situations, often we find problems which are written out in words instead of mathematical notation. In order to solve or simplify these types of problems, we must change the words into the proper notation. The following table provides four steps to follow when changing problems from words into mathematical notation.

CHANGING WORDS TO SYMBOLS

1. Identify the words that represent mathematical operation(s) in the problem. Cross out those words and replace them with the symbol used for that operation.
2. Identify the numbers in the problem. Cross out the words that represent numbers and replace them with the digit used for the number.
3. Use the symbols and digits to write the mathematical expression or equation. (Pay close attention to the wording in the problem to ensure that you write the symbols and digits in the proper order.)
4. Simplify or solve as necessary.

Below are lists of words that represent the arithmetic operations of addition and subtraction.

<u>ADDITION</u>	<u>SUBTRACTION</u>
add	subtract
sum	subtracted ***from****
plus	minus
total	difference
increase	less
more	less ***than****
more ***than****	decrease
combined	reduce
altogether	remain
in all	fewer
	(other comparison words)

the words* ***than* *and* ***from*** *will reverse the order of how the expression is written*

Example 20 **Find the sum of seven and negative eleven.**

+ 7 (−11)

Find the ~~**sum**~~ of ~~seven~~ and ~~negative eleven~~.

$$7+(-11)=\boxed{-4}$$

Example 21 **Find the difference between ten and fifteen.**

− 10 15

Find the ~~**difference**~~ between ~~ten~~ and ~~fifteen~~.

$$10-15=$$

$$10+(-15)=\boxed{-5}$$

In examples 22 and 23, pay close attention to how the word ***"than"*** changes the outcome of the problem.

Example 22 **Find six hundred fourteen less two hundred five.**

614 − 205

Find the ~~six hundred fourteen~~ ~~**less**~~ ~~two hundred five~~.

$$614-205=$$

$$614+(-205)=\boxed{409}$$

Example 23

Find six hundred fourteen less than two hundred five. *(The word* ***than*** *will reverse the order.)*

614 − 205

Find the ~~six hundred fourteen~~ **less** **than** ~~two hundred five~~

$$205 - 614 =$$

$$205 + (-614) = \boxed{-409}$$

Example 24

Find five more than negative eighteen. *(The word* ***than*** *will reverse the order.)*

5 + (-18)

Find ~~five~~ **more** **than** ~~negative eighteen~~.

$$-18 + 5 = \boxed{-13}$$

Example 25

Find the total of three hundred five, negative twelve and forty-seven.

+ 305 (-12) 47

Find the **total** of ~~three hundred five~~, ~~negative twelve~~, and ~~forty-seven~~.

$$305 + (-12) + 47 = \boxed{340}$$

Example 26

Subtract eight from negative seven. *(The word* ***from*** *will reverse the order.)*

− 8 (-7)

~~Subtract~~ ~~eight~~ **from** ~~negative seven~~.

$$(-7) - 8 =$$

$$(-7) + (-8) = \boxed{-15}$$

Example 27

Cadee has \$50.00 in her savings account. On Monday, she deposits \$25.00 into her account. On Wednesday, she withdraws \$32.00 from her account. On Friday, she deposits \$45.00 into her account. What is the balance in Cadee's account at the end of the week?

Cadee has \$50.00 in her savings account. On Monday, she ~~deposits~~ + \$25.00 into her account. On Wednesday, she ~~withdraws~~ − \$32.00 from her account. On Friday, she ~~deposits~~ + \$45.00 into her account.

$$\$50.00 + \$25.00 - \$32.00 + \$45.00 =$$

$$\$50.00 + \$25.00 + (-\$32.00) + \$45.00 = \boxed{\$88.00}$$

Example 28

Laura visits the local clothing store. She purchases a new shirt that costs \$12.00, a pair of jeans that cost \$23.00, a jacket that costs \$35.00 and a necklace that cost \$7.00. She has a coupon for \$5.00 off a total purchase of \$75.00 or more. What is her total bill before taxes?

First, determine if Laura can use the coupon. She must purchase at least \$75.00 in merchandise for the coupon to be valid. Find the sum of her purchases.

$$\$12.00 + \$23.00 + \$35.00 + \$7.00 = \$77.00$$

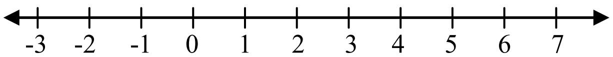
Her purchase is more than \$75.00, so the coupon is valid. Subtract the amount of the coupon to find the total bill before taxes.

$$\$77.00 - \$5.00 = \boxed{\$72.00}$$

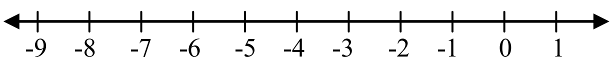
1.5 EXERCISES

In 1-6, use the given number line to simplify the following.

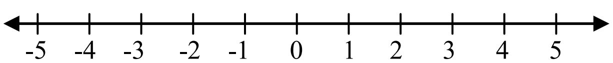
1. $3+4$



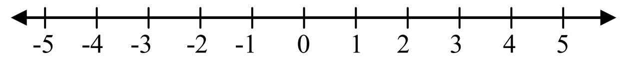
2. $(-2)+(-7)$



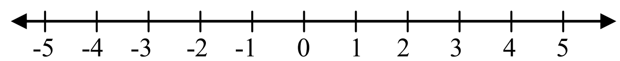
3. $5+(-8)+2$



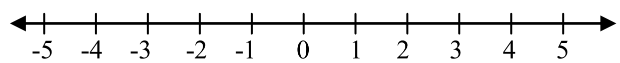
4. $2-6$



5. $0-5$



6. $4-1-(-2)$



In 7-22, simplify the following.

7. $8+(-3)$
8. $71+28$
9. $(-7)-(-3)$
10. $(-5)-8$
11. $4-(-6)$
12. $(-6)+(-2)$
13. $30-45$
14. $(-12)+3$
15. $129+(-37)$
16. $(-6)-(-6)$
17. $(-54)-31$
18. $(-10)+(-5)-12$
19. $(-6)-(-8)+(-12)-7$
20. $93+(-12)-34$
21. $(-3)+4-(-23)-10$
22. $(-10)+14+25-16$

In 23-32, simplify the following.

23. Find the sum of five and negative ten.

24. Find two increased by seven.

25. Find negative eight more than three.

26. Combine four and negative five using addition.

27. Find the total of negative two, three, and negative one.

28. Find the difference between negative twenty and negative three.

29. Subtract eighteen from negative twenty-one.

30. Find negative eleven subtract two.

31. Find twelve less than negative ten.

32. Find twelve less negative ten.

In 33-37, solve the following application problems.

33. Mike has \$125.00 in his checking account. He writes a check for \$115.00, makes a deposit of \$43.00, and writes another check for \$57.00. Find the balance of his checking account.

34. The temperature on a February day is negative six degrees Celsius in the morning. If the temperature drops three degrees by 7:00 a.m., raises four degrees before 8:00 a.m., and drops seven degrees before 9:00 a.m. Find the temperature at 9:00 a.m.

35. Sharlie has \$250.00 to spend on her books and school supplies this semester. Her Math book costs \$63.00, her English book costs \$52.00, her Biology book costs \$85.00, and her calculator costs \$22.00. She also purchases pencils, pens, paper and folders that cost a total of \$13.00. How much money will Sharlie have remaining?

36. The average temperature on the surface of the Earth is fifteen degrees Celsius. The average temperature on the surface of Mars is negative sixty-three degrees Celsius. How many degrees warmer is the surface of the Earth than the surface of Mars?

37. Hailey went into the local office supply store. There she purchased a cellular telephone charger for \$30.00 with a \$13.00 mail-in rebate. She also purchased a box of compact discs for \$18.00 with a \$5.00 mail-in rebate. Finally, she purchased a box of folders for \$10.00 with a \$3.00 mail-in rebate. What was the total cost of Hailey's purchases after the mail-in rebates?

1.6 Multiplication of Integers

Multiplication is the third of the four basic arithmetic operations and is the process by which a number is added a given number of times. For example, five times two means 2 +2 + 2 + 2 + 2, or two added five times. Multiplication is a shorthand way of writing repeated addition.

There are several symbols used for multiplication. They are:

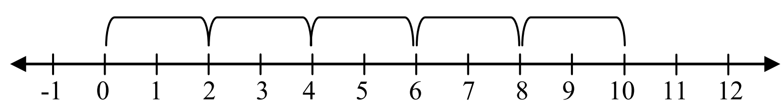
1. $\times$ which may be used in arithmetic, as 4×3. (*In this text, the symbol* $\times$ *will not be used to indicate multiplication. It tends to be confused when used with letters of algebra.)*
2. $\cdot$, a dot, placed above the line between two numbers, as $4\cdot3$.
3. No sign between letters, as *abc.*
4. Parentheses, or other symbols of grouping, as $(4)(3)$.

There are three parts to a multiplication problem. They include the multiplicand, the multiplier, and the product. The number that is being multiplied is called the **multiplicand**. (The multiplicand tells us the number of times we repeat a number) The number by which the multiplicand is multiplied is called the **multiplier**. (The multiplier is the number that is repeated.) The multiplicand and multiplier are known as factors. The result obtained in multiplication is called the **product**.

$$(multiplicand)\cdot(multiplier) = product$$

The following diagram illustrates how to find the product of five times two.

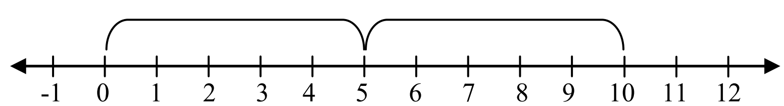
$$5\cdot2 = \underbrace{2+2+2+2+2}_{\textit{5 times}} = 10$$



Notice:

$$2\cdot5 = 10$$

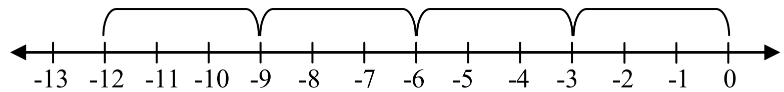
$$\underbrace{5+5}_{\textit{2 times}} = 10$$



The product of $2\cdot5$ is the same as the product of $5\cdot2$. This concept is known as the Commutative Property of Multiplication which was introduced in Section 1 of this chapter.
Recall that the Commutative Property of Multiplication states the order in which two numbers are multiplied does not change the product.

Example 1 **Find the product of $4\cdot(-3)$.**

$$(-3)+(-3)+(-3)+(-3)=\boxed{-12}$$

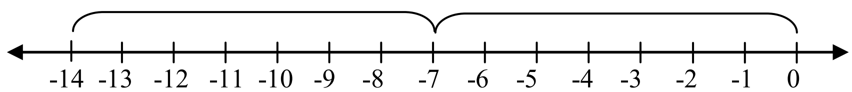


Example 2 **Find the product of $(-7)\cdot 2$.**

$$2\cdot(-7)$$

You cannot have negative seven times. Use the Commutative Property of Multiplication to change the order.

$$(-7)+(-7)=\boxed{-14}$$



In examples 1 and 2, it was found that the product of a positive number and a negative number produces a negative number. What happens when a negative number is multiplied by a negative number? Look at the following pattern to answer this question.

$$(-2)\cdot 3=-6$$

$$(-2)\cdot 2=-4$$

Notice, the products increase by 2 each step down the list.

$$(-2)\cdot 1=-2$$

$$(-2)\cdot 0=0$$

The numbers on the left of the equal sign decrease by 1 for each step down the list. The products on the right increase by 2 for each step down the list. To continue the pattern $-2\cdot(-1)$ should be 2 more than $-2\cdot 0$, or 2 more than 0, so :

$$(-2)\cdot(-1)=2$$

The pattern continues with:

$$(-2)\cdot(-1)=2$$

$$(-2)\cdot(-2)=4$$

$$(-2)\cdot(-3)=6$$

In the above example, it is shown that the product of two negatives produces a positive answer. The saying “A negative times a negative equals a positive” will help you remember this concept.

The multiplication of integers chart located below will aid in identifying the sign of a given product.

MULTIPLICATION OF INTEGERS

like signs
- positive · positive = positive
- negative · negative = positive

unlike signs
- positive · negative = negative
- negative · posititve = negative

Understanding that multiplication is a shorthand way of writing repeated addition is important but, we do not want to write repeated addition with every multiplication problem. Therefore, it is imperative to know the basic multiplication facts. The following table can help identify the products of the integers from 1 to 12. To use the table, pick the two numbers that you want to multiply. Find the first number in the gray area on the left side of the table. Find the second number in the gray area across the top of the table. The product of the two numbers will be located by extending the row where the first number lies and the column where the second number lies to the location where they meet.

MULTIPLICATION FACTS TABLE

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	18	27	36	45	54	63	72	81	90	99	108
10	10	20	30	40	50	60	70	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

Memorizing the multiplication facts listed on the table above will help you become successful in Pre Algebra and future mathematics studies.

Example 3 **Find the product of the following.**

a) $6\cdot(-4)=\boxed{-24}$

b) $9\cdot 7=\boxed{63}$

c) $(-12)\cdot 3=\boxed{-36}$

d) $(-10)\cdot 0=\boxed{0}$

e) $(-5)\cdot(-7)=\boxed{35}$

f) $6\cdot(-3)=\boxed{-18}$

When finding the product of more than two numbers, the order in which you multiply does not matter. (The Associative Property of Multiplication). The following examples illustrate this concept.

Example 4 **Find the product of $4(-5)(-2)$.**

$$\underbrace{4(-5)}(-2)$$
$$(-20)(-2)$$
$$\boxed{40}$$

$$4\underbrace{(-5)(-2)}$$
$$4(10)$$
$$\boxed{40}$$

Example 5 **Find the product of the following.**

a)
$$\underbrace{(-3)(-1)}(-11)$$
$$(3)(-11)$$
$$\boxed{-33}$$

b)
$$\underbrace{(-2)(3)}(5)$$
$$(-6)(5)$$
$$\boxed{-30}$$

c)
$$\underbrace{(-1)(-4)}\underbrace{(-3)(-2)}(4)$$
$$\underbrace{(4)\,(6)}\,(4)$$
$$(24)(4)$$
$$\boxed{96}$$

Note: If you have an odd number of negatives, your product will be negative.
If you have an even number of negatives, your product will be positive.

To find the product of two larger numbers, use the basic multiplication facts and the stacking method. To use the stacking method, start by writing, or *stacking*, one number above the other number. Line up the numbers so that the last digit in the first number is directly over the last digit in the second number. Draw a line under the bottom number. Multiply. (In the next three examples, pay close attention to the wording to the right of the problems. This information will help you through the process of multiplying larger numbers.)

Example 6 **Find the product of** $341 \cdot 2$.

$$\begin{array}{r} 341 \\ \times\ 2 \\ \hline \end{array}$$

Stack one number above the other so that the ones' place digits are lined up. Draw a line under the bottom number.

$$\begin{array}{r} 341 \\ \times\ 2 \\ \hline 2 \end{array}$$

Multiply both numbers in the one's place value ($2 \cdot 1 = 2$). Place the product, 2, below the line in the ones' place column.

$$\begin{array}{r} 341 \\ \times\ 2 \\ \hline 82 \end{array}$$

Multiply the digit in the tens' place column (4) by the second number (2). The result is $4 \cdot 2 = 8$. Place the product, 8, below the line in the tens' place column.

$$\begin{array}{r} 341 \\ \times\ 2 \\ \hline 682 \end{array}$$

Multiply the digit in the hundreds' place column (3) by the second number (2). The result is $3 \cdot 2 = 6$. Place the product, 6, below the line in the hundreds' place column.

$341 \cdot 2 = \boxed{682}$

State your answer.

Example 7

Find the product of $59 \cdot 7$.

$$\begin{array}{r} 59 \\ \times 7 \\ \hline \end{array}$$

Stack one number above the other so that the ones' place digits are lined up. Draw a line under the bottom number.

$$\begin{array}{r} \overset{6}{5}9 \\ \times 7 \\ \hline 3 \end{array}$$

Multiply the numbers in the one's place value ($9 \cdot 7 = 63$). Because the product of these numbers has two digits, place the digit in the ones place, 3, below the line in the ones' place column. Carry the digit in the tens' place, 6, above the digit in the tens place.

$$\begin{array}{r} \overset{6}{5}9 \\ \times\ 7 \\ \hline 413 \end{array}$$

Multiply the digit in the tens' place column (5) by the second number (7). The result is $5 \cdot 7 = 35$. Add the 6 that you carried to 35. ($35 + 6 = 41$) and place this answer below the line. The 1 goes in the tens' place column and the 4 in the hundreds' place column.

$59 \cdot 7 = \boxed{413}$

State your answer.

Example 8

Find the product of $(-43)(26)$.

$$\begin{array}{r} \overset{1}{4}3 \\ \times 26 \\ \hline 258 \end{array}$$

Stack one number above the other so that the ones' place digits are lined up. Draw a line under the bottom number. Multiply the numbers in the ones' place values $(6 \cdot 3 = 18)$. Because the product of these numbers has two digits, place the digit in the ones place, 8, below the line in the ones' place column. Carry the digit in the tens' place, 1, above the digit in the tens place.

$$\begin{array}{r} 43 \\ \times 26 \\ \hline 258 \\ 86 \\ \hline \end{array}$$

Multiply the digit in the tens' place column (4) by the ones' place digit in the second number (6). The result is $6 \cdot 4 = 24$. Add the 1 to the 24 (24 + 1 = 25) and place the answer below the line and to the left of the 8.

Now repeat this process with the tens place value in the second number.

$$\begin{array}{r} \overset{1}{4}3 \\ \times 26 \\ \hline \overset{1}{2}58 \\ +86 \\ \hline 1118 \end{array}$$

Find the sum.

$(-43)(26) = \boxed{-1{,}118}$

CHANGING WORDS TO SYMBOLS

As with addition and subtraction, you will come across mathematical problems involving multiplication that are written in words and need to be translated into mathematical notation. Use the same process of identifying the words that represent the operations and numbers in the problem, crossing them out, and replacing them with the symbol or digit as appropriate. Refer to the Changing Words to Symbols table provided in Section 1.5.

Below is a list of words that represent the arithmetic operation of multiplication.

<u>MULTIPLICATION</u>

Multiply
Product
Times
Part Of
Twice
Area
Volume

Example 9

Find the product of two and eight.

• 2 8

Find the ~~**product**~~ of ~~two~~ and ~~eight~~.

$$2\cdot 8=\boxed{16}$$

Example 10

Twice negative thirty-three multiplied by seventeen.

2 • (−33) • 17

~~**Twice**~~ ~~negative thirty-three~~ ~~**multiplied**~~ by ~~seventeen~~.

$$\underbrace{2(-33)}\cdot(17)$$

$$(-66)\cdot(17)$$

$$\boxed{-1{,}122}$$

Example 11 **One-half of negative eight times negative seven.**

$$\frac{1}{2} \quad \bullet \quad (-8) \quad \bullet \quad (-7)$$

~~One half~~ ~~**of**~~ ~~negative eight~~ ~~**times**~~ ~~negative seven~~.

$$\frac{1}{2}\cdot(-8)\cdot(-7)$$

$$(-4)\cdot(-7)$$

$$\boxed{28}$$

Example 12 **Multiply five, negative two, four, and negative one.**

$$\bullet \quad 5 \quad (-2) \quad 4 \quad (-1)$$

~~**Multiply**~~ ~~five~~, ~~negative two~~, ~~four~~ and ~~negative one~~.

$$5\cdot(-2)\cdot 4\cdot(-1)$$

$$(-10)(-4)$$

$$\boxed{40}$$

Example 13 **A compact disk (CD) can hold 650 megabytes (MB) of information. How many megabytes can 17 disks hold?**

$17\cdot 650$ — *If one CD holds 650 MB of information then 17 CD's will hold 17 times 650 MB.*

$$\begin{array}{r} \overset{3}{6}\,5\,0 \\ \times 1\,7 \\ \hline \overset{1}{4}\,5\,5\,0 \\ +6\,5\,0 \\ \hline 1\,1\,0\,5\,0 \end{array}$$

To find the product, stack the larger number above the smaller number. Multiply and add as outlined in examples 6 through 8.

State your answer.

17 compact disks will hold $\boxed{\textbf{11,050 megabytes}}$ of information.

1.6 EXERCISES

In 1-20, simplify the following.

1. $(-4)9$
2. $(11)(7)$
3. $43(19)$
4. $(1,243)(-35)$
5. $6(-5)(-2)$
6. $3\cdot(-2)\cdot(-7)$
7. $0(14)$
8. $21\cdot 32$
9. $101\cdot 99$
10. $(-4)(-12)(-30)$
11. $(-7)(-2)$
12. $10\cdot(-5)\cdot 0$
13. $(-2)\cdot 3\cdot(-7)$
14. $59\cdot 7$
15. $(-9)\cdot 6$
16. $(-20)(5)(2)$
17. $(-1)(2)(7)(-3)$
18. $365\cdot 8$
19. $72\cdot 15$
20. $(63)(-2)(-1)(5)$

In 21-26, simplify the following.

21. Find the product of two and four.
22. Multiply two, four, negative three, and one.
23. Twice negative two multiplied by eight.
24. Three times the product of negative two and one.
25. The product of fifty-seven and thirty-six.
26. Multiply twelve, negative thirty-two, and negative two.

In 27-32, solve the following application problems.

27. A weather forecaster predicts that the temperature will drop five degrees each hour for the next six hours. Represent this drop as a product of integers and find the total drop in temperature.

28. Joe lost $400.00 on each of seven consecutive days in the stock market. Represent his total loss as a product of integers and find his total loss.

29. An average cow eats three pounds of grain per day. Find the amount of grain a cow eats in one year.

30. A line of print on a computer contains eighty characters (spaces, letters, numbers, punctuation, etc.). How many characters are on twenty-eight lines?

31. A rectangular house measures forty-four feet by sixty-two feet. Find the square feet of the house by multiplying the lengths of the sides together.

32. The seats in a lecture hall are arranged in fifteen rows with eight seats in each row. Find the number of seats in the lecture hall.

1.7 Division of Integers

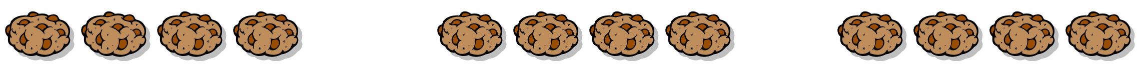
Division is the last of the four basic arithmetic operations and is the process of separating quantities into equal parts**.** The following situation illustrates the concept of division.

Suppose we have twelve cookies to divide evenly among four people. How many cookies will each person get?

There are four groups of three, so each person will get three cookies.

$$12 \div 4 = 3$$

Now, suppose that one person did not show up. How many cookies will each person get if we divide the twelve cookies evenly among three people?



There are three groups of four, so each person will get four cookies.

$$12 \div 3 = 4$$

There are three parts to a division problem. They include the dividend, the divisor, and the quotient. The **dividend** is the number that is to be divided. The **divisor** is the number by which a given number is divided. The **quotient** is the result when one number is divided by another number. A division problem can be written three different ways:

$$\frac{dividend}{divisor} = quotient \qquad (dividend) \div (divisor) = quotient \qquad divisor\overline{\smash{)}dividend}^{\;quotient}$$

Division is the opposite or inverse operation of multiplication. In fact, division can be defined in terms of multiplication. The definition for division in general terms: For all real number *a*, *b*, and *c* where $b \neq 0$, $a \div b = c$ if and only if $c \cdot b = a$. For example: $12 \div 4 = 3$ if and only if $3 \cdot 4 = 12$. The division of integers chart located below will aid in identify the sign of a given quotient. Notice that the rules for multiplication and division of integers are the same. If you multiply or divide two numbers with like signs their product or quotient is positive. If you multiply or divide two numbers with unlike signs their product or quotient is negative.

DIVISION OF INTEGERS

like signs	positive $\div$ positive = positive
	negative $\div$ negative = positive
unlike signs	positive $\div$ negative = negative
	negative $\div$ posititve = negative

Example 1

Find the quotient of $100 \div (-25)$ **.**

$100 \div (-25) = \boxed{-4}$

The signs are different so the quotient is negative.

Example 2

Find the quotient of $(-132) \div (-11)$ **.**

$(-132) \div (-11) = \boxed{12}$

The signs are the same so the quotient is positive.

Example 3

Find the quotient of $(-215) \div 5$ **.**

$(-25) \div 5 = \boxed{-5}$

The signs are different so the quotient is negative.

Example 4

Find the quotient of $120 \div (-2) \div (-3)$ **.**

$12 \div (-2) \div (-3)$

$(-6) \div (-3)$

$\boxed{2}$

Division is not commutative so we must work left to right. The signs are different so the first quotient is negative.

The signs are the same so the second quotient is positive.

Example 5

Find the quotient of $(-360) \div 12 \div 5$ **.**

$(-36) \div 6 \div 3$

$(-6) \div 3$

$\boxed{-2}$

Division is not commutative so we must work left to right. The signs are different so the first quotient is negative.

The signs are the different again, so the second quotient is negative

When working with division, recall the properties of division into zero and division by zero introduced in 1.1.

> **DIVISION INTO ZERO PROPERTY**
>
> For all real numbers *a*, $a \neq 0$,
>
> $$0 \div a = \frac{0}{a} = 0$$

The Division Into Zero Property states that when any non-zero real number is divided into zero the quotient will equal zero.

> **DIVISION BY ZERO PROPERTY**
>
> For all non-zero real numbers *a,*
>
> $$a \div 0 = \frac{a}{0} = \text{undefined}$$

Why is division of non-zero numbers undefined? Let's look at the example $12 \div 0$. What do you think the answer should be? Did you think it was 12 or maybe 0? Using the definition of division would either of these two answers be correct? Let's start with 12.

$12 \div 0 = 12$ *if and only if* $12 \cdot 0 = 12$ — We know that $12 \cdot 0 = 0$ and $0 \neq 12$, so this answer could not be correct.

What about 0?

$12 \div 0 = 0$ *if and only if* $0 \cdot 0 = 12$ — We know that $0 \cdot 0 = 0$ and $0 \neq 12$, so this answer could not be correct.

If you continue to try values, you will never find a number that works. Therefore, **division of nonzero numbers by 0 is always undefined.**

Example 6

Find the quotient of the following.

a) $35 \div 0$

$35 \div 0 = \boxed{\textit{undefined}}$

b) $0 \div 35$

$0 \div 35 = \boxed{0}$

Example 7

Find the quotient of the following.

a) $0 \div (-200) \div 10$

$0 \div (-200) \div 10 = \boxed{0}$

b) $18 \div 0 \div 3$

$18 \div 0 \div 3 = \boxed{\textit{undefined}}$

c) $(-52) \div 4 \div 0$

$(-13) \div 0$

$(-13) \div 0 = \boxed{\textit{undefined}}$

d) $0 \div (-24) \div 8$

$0 \div (-24) \div 8 = \boxed{0}$

The procedure known as long division is used when dividing multi-digit numbers. Long division breaks down a division problem into a series of easier steps. As in all division problems, one number, called the dividend, is divided by another, called the divisor, producing a result called the quotient. It enables computations involving arbitrarily large numbers to be performed by following a series of simple steps. The following two examples illustrate long division in a step-by-step format.

Example 8 **Simplify using long division.** $512 \div 4$

Step 1: Rewrite as a long division problem.	$4\overline{)512}$
Step 2: a) 5 divided by 4 = 1 with 1 remaining. Place the **1** above the 5.	1 $4\overline{)512}$
b) 1 multiplied by 4 = 4. Place the **4** below the 5.	1 $4\overline{)512}$ 4
c) Subtract.	1 $4\overline{)512}$ $\ominus 4$ $\overline{1}$
Step 3: Bring down the next digit.	1 $4\overline{)512}$ $\ominus 4\downarrow$ $\overline{11}$
Step 4: Repeat steps 2 and 3 until we have brought down the last digit.	128 $4\overline{)512}$ $\ominus 4\downarrow$ $\overline{11}$ $\ominus 8\downarrow$ $\overline{32}$ $\ominus 32$ $\overline{0}$
Step 5: State your answer.	128
Step 6: Check. Multiply the quotient by the divisor.	$128 \cdot 4 = 512$

$$512 \div 4 = \boxed{128}$$

Example 9 **Simplify using long division.** $4140 \div (-12)$

Step 1: Rewrite as a long division problem. **Our quotient will be **negative** because a positive divided by a negative is negative. The sign with be included in the answer.*	$12\overline{)4140}$
Step 2: a) 41 divided by 12 = 3 with 5 remaining. Place the **3** above the 41.	3 $12\overline{)4140}$
b) 12 multiplied by 3 = 36. Place the **36** below the 41.	3 $12\overline{)4140}$ 36
c) Subtract.	3 $12\overline{)4140}$ $\ominus\ \underline{36}$ 5
Step 3: Bring down the next digit.	3 $12\overline{)4140}$ $\ominus\ \underline{36}\downarrow$ 54
Step 4: Repeat steps 2 and 3 until we have brought down the last digit.	345 $12\overline{)4140}$ $\ominus\ \underline{36}\downarrow$ 54 $\ominus\ \underline{48}\downarrow$ 60 $\ominus\ \underline{60}$ 0
Step 5: State your answer. (*Remember the sign!)	-345
Step 6: Check. Multiply the quotient by the divisor.	$-345 \cdot (-12) = 4{,}140$

$$4140 \div (-12) = \boxed{-345}$$

Example 10

Use long division to simplify. $(-15,600)\div(-25)$

```
      6 2 4
25)15,6 0 0
 ⊖1 5 0
     6 0
   ⊖5 0
     1 0 0
   ⊖1 0 0
         0
```

Rewrite as a long division problem. The quotient will be ***positive*** *because a negative divided by a negative is a positive.*

Divide using the steps outlined in examples 8 and 9.

$(-15,600)\div(-25)=\boxed{624}$

State your answer.

Check: $624\cdot(-25)=-15,600$

Check by multiplication.

Example 11

Use long division to simplify. $(-189,527)\div(13)$

```
      1 4,5 7 9
13)1 8 9,5 2 7
 ⊖1 3
     5 9
   ⊖5 2
       7 5
     ⊖6 5
       1 0 2
      ⊖ 9 1
         1 1 7
       ⊖1 1 7
             0
```

Rewrite as a long division problem. The quotient will be ***negative*** *because a negative divided by a positive is a negative.*

Divide.

$(-189,527)\div(13)=\boxed{-14,597}$

State your answer.

Check: $-14,597\cdot 13=-189,527$

Check by multiplication.

CHANGING WORDS TO SYMBOLS

As with addition, subtraction, and multiplication, we will come across mathematical problems involving division that are written in words and need to be translated into mathematical notation. Use the same process of identifying the words that represent the operations and numbers in the problem, crossing them out, and replacing them with the symbol or digit as appropriate. Refer to the Changing Words to Symbols table provided in Section 1.5.

Below is a list of words that represent the arithmetic operation of division.

<u>DIVISION</u>

Divide
Quotient
Divided By
Divided *Into**
Split
Each
Shared
Per
Equal Parts
Average
Ratio Of

the word **into will reverse the order of how the expression is written.*

Example 12 **Find the quotient of ninety-nine and negative eleven.**

$\div$ 99 (-11)

Find the **~~quotient~~** of ~~ninety-nine~~ and ~~negative eleven~~.

$$99 \div (-11) = \boxed{-9}$$

Example 13 **Split fifty-six in eight equal parts.**

$\div$ 56 8

~~Split~~ ~~fifty-six~~ in ~~eight~~ equal parts.

$$56 \div 8 = \boxed{7}$$

Example 14 **Find negative nine hundred thirty-six divided by twelve.**

-936 $\div$ 12

Find ~~negative nine hundred thirty-six~~ **~~divided by~~** ~~twelve~~.

$$(-936) \div 12 = \boxed{-78}$$

Example 15

Divide negative six into negative seventy-two. *(The word **into** will reverse the order.)*

$\div \quad (-6) \longleftrightarrow (-72)$

~~**Divide**~~ ~~negative six~~ **into** ~~seventy-two~~.

$$(-72) \div (-6) = \boxed{12}$$

Example 16

Mrs. Walker has thirty-two students in her kindergarten class. She bought two hundred eighty-eight stickers to give to her students for their birthday. How many stickers will each student receive on their birthday?

To find the number of stickers each student will receive, divide the total number of stickers by the number of students.

$$288 \div 32 = 24$$

Each student will receive **24 stickers.**

Example 17

Trevor, Cody, and Bradly share a paper route to earn money for their college expenses. Last summer they earned two thousand, nine hundred thirteen dollars. How much money did each of them earn?

To find the amount of money each person earned, divide the total amount earned by the number of people.

$$\$2,913.00 \div 3 = \$971$$

Trevor, Cody, and Bradly each earned **$971.00.**

1.7 EXERCISES

In 1-20, simplify the following.

1. $9 \div 3$
2. $\frac{12}{4}$
3. $0 \div 5$
4. $\frac{(-36)}{3}$
5. $(-45) \div (-9)$
6. $2 \div 0$
7. $\frac{(-49)}{(-7)}$
8. $\frac{18}{3}$
9. $5{,}781 \div 123$
10. $480 \div (-8)$
11. $\frac{56}{8}$
12. $6\overline{)222}$
13. $240 \div (-40)$
14. $16 \div 16$
15. $0 \div 215$
16. $100 \div 5 \div 2$
17. $715 \div 55$
18. $85 \div 5$
19. $7 \div 0$
20. $225 \div 25 \div 3$

In 21-26, simplify the following.

21. Find the quotient of sixty-six and three.
22. Divide two hundred twenty-five by negative fifteen.
23. Divide negative three into twenty-seven.
24. Find the ratio of eighteen and three.
25. Find the quotient of one thousand, fourteen and three.
26. Split seventy-two in eight equal parts.

In 27-30, solve the following application problems.

27. A card player had a score of negative twelve for the total of four rounds in a game. Find her average score for each round.

28. An eighteen-hole golf course is five thousand, five hundred eighty yards long. If the distance to each hole is the same, find the distance between holes.

29. How many yards are in one mile? (A mile is five thousand, two hundred eighty feet and a yard is three feet.)

30. Twenty-one people pooled their money together to buy lottery tickets. One of the tickets won a prize of five million, two hundred ninety two thousand dollars. How much does each person receive?

CHAPTER 1 REVIEW EXERCISES

1. What is the difference between a whole number and a natural number?
2. What are the differences between real numbers and integers?
3. If you multiply by zero, what is the product?
4. If you divide by zero, what is the quotient?

In 5-6, write the following numbers in words.

5. 17.2 ____
6. –3,247,000.386 ____________________________________

In 7-8, change from words into digits.

7. One hundred forty-three and nine hundredths ______________________
8. Two thousand, eleven and four hundred fifty-eight hundred-thousandths______________

In 9-12, Compare the following numbers. Insert $>$, $<$, or $=$ ***between the pair of numbers.***

9. 3.12 ____ 3.1
10. $-(-7)$ ____ $-|-7|$
11. $-|-2|$ ____ $-|2|$
12. $|3|$ ____ (-3)

In 13-16, complete the following table by rounding to the indicated place value.

	Number	Thousands	Hundreds	Tens	Ones	Tenths	Hundredths	Thousandths
13.	13.543							
14.	762.48							
15.	-14.095							
16.	3,457.2834							

In 17-42, simplify the following.

17. $|-7.5|$

18. $\left(-\frac{3}{4}\right)$

19. $-|1|$

20. $-(-6)$

21. $-\left|-\frac{1}{3}\right|$

22. $(-214)+47$

23. $15-17$

24. $1-3-(-4)$

25. $(-25)-17+39-0$

26. Find fourteen less than seven.

27. Find eight increased by three.

28. Subtract one and negative four.

29. Find negative two more than three.

30. $(-7)(-8)$

31. $\frac{(-108)}{12}$

32. $(-3)(-6)(-1)(10)$

33. $111\div 3$

34. $4\cdot 2\cdot(-1)$

35. $13\div 0$

36. $\frac{0}{(-4)}$

37. $150\div(-3)\div(2)$

38. Divide five thousand, ninety-two by four.

39. Find the product of negative twelve and eleven.

40. Find the quotient of negative seventy-two and negative twelve.

41. Multiply three by fourteen.

42. Divide zero into five.

In 43-50, solve the following application problems.

43. Tanner has $168.00 in his checking account. On Monday, he makes a deposit of $25.00. On Tuesday, he writes a check for $37.00. On Wednesday, he writes a check for $74.00. On Friday he writes a check for $6.00. What is the balance in Tanner's checking account at the end of the week?

44. Natalie spent $4.00 for her lunch five days in a row. What is the total amount Natalie spent for her lunches during the five-day period?

45. Beckie Hermansen teaches American Sign Language classes for fifty-five dollars per student for a seven week session. She collected one thousand, four hundred thirty dollars from her students this session. How many students are in the class?

46. A truck hauls wheat to a storage granary. It carries a total of five thousand, eight hundred ten bushels of wheat in fourteen trips. If the truck hauls an equal amount of weight each time, how much does the truck haul per trip?

47. An apartment building has three floors. Each floor has five rows of apartments with four apartments in each row. How many apartments are there in the building?

48. One tablespoon of olive oil contains one hundred twenty-five calories. How many calories are in three tablespoons of olive oil?

49. Lance scored 85, 91, 100, 63, 77, and 73 on his science tests. Round each score to the nearest tens and find the total points he earned on the tests.

50. An appliance store advertises three refrigerators on sale for $799.00, $1,299.00, and $999.00. If you were to purchase all three refrigerators, what would the total bill be?

CHAPTER 2

PROPERTIES OF NUMBERS

2.1 Divisibility & Prime Factorization

In Chapter 1, the types of numbers and their basic mathematical operations were discussed. This chapter will build upon those ideas and introduce additional definitions and operations dealing with the set of real numbers. The first concept to be examined is how to find factors of numbers. A **factor** is a number that divides into a whole number with a remainder of zero. Factors can also be thought as the numbers that are multiplied together to obtain a given product. (A product is the answer to a multiplication problem.) For example, 3 and 5 are factors of 15. 1 and 15 are also factors of 15.

The first step in finding factors of whole numbers is to determine when a whole number can be divided evenly by another whole number. A whole number is **divisible** by another number if the remainder is 0 when the first number is divided by the second. The divisibility rules for 1, 2, 3, 4, 5, 6, 9, and 10 are stated in the figure 2.1.

A whole number is **even** if it is divisible by 2. A whole number is **odd** if it is not divisible by 2.

DIVISIBILITY TESTS

A whole number is divisible by:

- **1 all** numbers are divisible by 1.
- **2** if the ones digit is divisible by 2, or the number is even.
- **3** if the sum of its digits is divisible by 3.
- **4** if the number formed by the last two digits is divisible by 4.
- **5** if the ones digit is 0 or 5.
- **6** if the number is divisible by both 2 and 3.
- **9** if the sum of the digits is divisible by 9.
- **10** if the ones digit is 0.

Example 1

Tell whether 138 is divisible by 1, 2, 3, 4, 5, 6, 9, or 10. Then classify it as *even* or *odd*.

1: Yes; all numbers are divisible by 1.
2: Yes; the ones digit, 8, is divisible by 2.
3: Yes; the sum of the digits, $1+3+8=12$, is divisible by 3.
4: No; the number formed by the last two digits, 38, is not divisible by 4.
5: No; the ones digit, 8, is not a 0 or 5.
6: Yes; the number is divisible by both 2 and 3.
9: No; the sum of the digits, $1+3+8=12$, is not divisible by 9.
10: No; the ones digit is not a 0.

The number 138 is divisible by **1**, **2**, **3**, and **6** and is **even**.

Example 2

Tell whether 63,225 is divisible by 1, 2, 3, 4, 5, 6, 9, or 10. Then classify it as *even* or *odd*.

1: Yes; all numbers are divisible by 1.
2: No; the ones digit, 5, is not divisible by 2.
3: Yes; the sum of the digits, $6+3+2+2+5=18$, is divisible by 3.
4: No; the number formed by the last two digits, 25, is not divisible by 4.
5: Yes; the ones digit is a 5.
6: No; the number is not divisible by both 2 and 3.
9: Yes; the sum of the digits, $6+3+2+2+5=18$, is divisible by 9.
10: No; the ones digit is not a 0.

The number 63,225 is divisible by **1**, **3**, **5**, and **9** and is **odd**.

Example 3

A bakery requires 3 pounds of flour for a loaf of bread. If they have 38 pounds of flour, can they use all the flour for bread?

Use divisibility rules to check whether 38 is divisible by 3. If it is divisible by 3 then the bakery can use all the flour for bread.

38 is not divisible by three because the sum of the digits, 3 + 8 = 11, is not divisible by three.

No, the bakery cannot use all the flour for bread.

A **prime number** is a whole number greater than one with exactly two unique factors, one and itself. The prime numbers to 100 include: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, …

A **composite number** is a whole number greater than one that has more than two factors.

The numbers 0 and 1 are neither prime nor composite. The number 0 has an infinite number of factors. For instance, $0\cdot1=0$, $0\cdot2=0$, $0\cdot3=0$, $0\cdot4=0$, ... the numbers 1, 2, 3, 4, … are all factors of 0. One is not a prime or composite number because it's only factor is 1.

Example 4

Determine whether the following numbers are *prime, composite, or neither.*

a) **28** ***composite***

The factors of 28 are 1 and 28, 2 and 14, and 4 and 7. Since 28 has more than two factors, it is a composite number.

b) **1** ***neither***

1 is the only factor of 1, so it is neither prime nor composite.

c) **17** ***prime***

The factors of 17 are 1 and 17. Since there are exactly two factors, 1 and the number itself, 17 is a prime number.

Every number greater than one has at least two factors: the number one and the number itself. Some numbers have more factors than one and itself. The divisibility rules can help us to find all the factors of a given number.

Example 5 **Find all positive factors of 36.**

To find all the factors 36, start with the pair of factors that all numbers greater than one have, one and itself.

$$36 = 1 \cdot 36$$

The divisibility rules can be used to help find additional pairs of factors. 2 goes into 36, 18 times, so 2 and 18 are a pair of factors of 36.

$$\begin{aligned} 36 &= 1 \cdot 36 \\ &= 2 \cdot 18 \end{aligned}$$

Continue trying more numbers, increasing by one each time. 3 goes into 36, 12 times, so 3 and 12 are a pair of factors of 36. 4 goes into 36, 9 times, so 4 and 9 are another pair of factors of 36.

$$\begin{aligned} 36 &= 1 \cdot 36 \\ &= 2 \cdot 18 \\ &= 3 \cdot 12 \\ &= 4 \cdot 9 \end{aligned}$$

36 is not divisible by 5, so that number doesn't work. Leave that one out and continue on to 6.

$$\begin{aligned} 36 &= 1 \cdot 36 \\ &= 2 \cdot 18 \\ &= 3 \cdot 12 \\ &= 4 \cdot 9 \\ &= 6 \cdot 6 \end{aligned}$$

36 is not divisible by 7 or 8. 36 is divisible by 9. It is not necessary to include the factors $9 \cdot 4$ to the table, because they are the same factors as the earlier entry: $4 \cdot 9$. Once you reach a point where your factors repeat, you are finished finding the factors of your number.

$$\begin{aligned} 36 &= 1 \cdot 36 \\ &= 2 \cdot 18 \\ &= 3 \cdot 12 \\ &= 4 \cdot 9 \\ &= 6 \cdot 6 \\ &= 9 \cdot 4 \end{aligned}$$

same factors ($4 \cdot 9$ and $9 \cdot 4$)

The factors of 36 are $\boxed{1, 2, 3, 4, 6, 9, 12, 18, \textit{and } 36}$.

Example 6

Find all positive factors of 12.

$$\begin{aligned} 12 &= 1 \cdot 12 \\ &= 2 \cdot 6 \\ &= 3 \cdot 4 \\ &= 4 \cdot 3 \end{aligned}$$

same factors ($3 \cdot 4$ and $4 \cdot 3$)

Start with the pair of factors, 1 and itself. Then find additional factors using the numbers 2, 3, 4…. Once you repeat a set of factors, you are finished.

The factors of 12 are $\boxed{1, 2, 3, 4, 6, \textit{and } 12}$.

Example 7

Find all positive factors of 29.

The number one and the number itself are always factors of the number. So 1 and 29 are factors. To find any remaining factors, use the divisibility tests. The number 29 does not pass any divisibility tests, it is a prime number.

The factors of 29 are $\boxed{1 \textit{ and } 29}$.

Example 8

Find all positive factors of 27.

$$\begin{aligned} 27 &= 1 \cdot 27 \\ &= 3 \cdot 9 \\ &= 9 \cdot 3 \end{aligned}$$

same factors ($3 \cdot 9$ and $9 \cdot 3$)

Start with the pair of factors, 1 and itself. Then find additional factors using the numbers 2, 3, 4…. Once you repeat a set of factors, you are finished.

The factors of 27 are $\boxed{1, 3, 9 \textit{ and } 27}$.

Example 9

Find all positive factors of 48.

$$\begin{aligned} 48 &= 1 \cdot 48 \\ &= 2 \cdot 24 \\ &= 3 \cdot 16 \\ &= 4 \cdot 12 \\ &= 6 \cdot 8 \\ &= 8 \cdot 6 \end{aligned}$$

same factors ($6 \cdot 8$ and $8 \cdot 6$)

Start with the pair of factors, 1 and itself. Then find additional factors using the numbers 2, 3, 4…. Once you repeat a set of factors, you are finished.

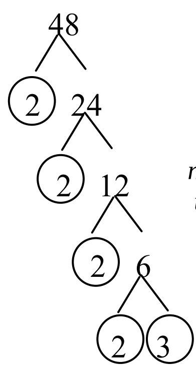
The factors of 48 are $\boxed{1, 2, 3, 4, 6, 8, 12, 16, 24 \textit{ and } 48}$.

Every composite number can be expressed as a product of prime numbers. This is called a **prime factorization** of the number. A **factor tree** can be used to find the prime factorization of a number**.**

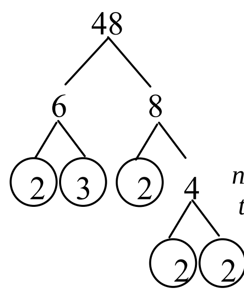
Example 10 Find the prime factorization of 48.

Start by writing the number that is being factored. Place two short diagonal lines below the number. Then choose any pair of whole number factors of the number and write those at the bottom of the two diagonal lines. Repeat the process with all composite numbers. (Remember to use the divisibility rules to help you find factors.)

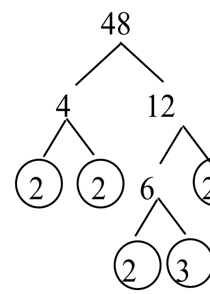
Prime factorization of the number 48 will be demonstrated in 3 ways to illustrate that the pairs of factors you begin with do not matter. You will end up with the same prime factors in the end. (Notice that the first pair of factors in the following prime factorizations of 48 are different.)



Circle the prime numbers and continue to factor any number that is not prime.



Circle the prime numbers and continue to factor any number that is not prime.



Except for the order, the prime factors (the numbers in circles) are the same. Because multiplication is commutative we can write the prime factorization arranged in order of size.

Therefore, the prime factorization of $\boxed{48 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3}$.

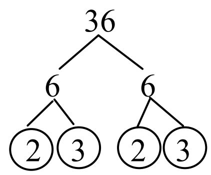
The fact that a composite number can be factored into prime factors in only one way is called the fundamental theorem of arithmetic.

FUNDAMENTAL THEOREM OF ARITHMETIC

Every composite number can be expressed uniquely as a product of primes, apart from the rearrangement of terms.

For example, 48 can be expressed as $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$. No other combination of prime factors (excluding different orderings of 2, 2, 2, 2, and 3) will yield 48. When using a factor tree, it does not matter what factors you to begin with. You will always end up with the same prime factors in the end.

Example 11 **Find the prime factorization of 36.**

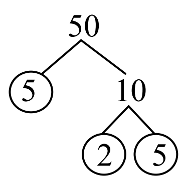


The prime factorization of $\boxed{36 = 2 \cdot 2 \cdot 3 \cdot 3}$.

Example 12 **Find the prime factorization of 47.**

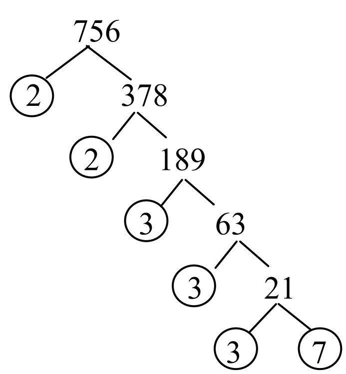
47 is a prime number so the prime factorization of $\boxed{47 = 47}$.

Example 13 **Find the prime factorization of 50.**



The prime factorization of $\boxed{50 = 2 \cdot 5 \cdot 5}$.

Example 14 **Find the prime factorization of 756.**



The prime factorization of $\boxed{756 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 7}$.

2.1 EXERCISES

***In 1-12, state whether each number is divisible by 2, 3, 4, 5, 6, 9, or 10. Then classify each number as* even *or* odd.**

1. 60
2. 138
3. 80
4. 83
5. 45
6. 605
7. 16,335
8. 27,453
9. 6,950
10. 900
11. 151,764
12. 5,203,570

***In 13-24, Determine whether the following numbers are* prime, composite, *or* neither.**

13. 12
14. 7
15. 29
16. 57
17. 0
18. 45
19. 13
20. 114
21. 1
22. 125
23. 179
24. 291

In 25-36, use a factor tree to find the prime factorization of each number.

25. 15
26. 12
27. 45
28. 40
29. 104
30. 80
31. 13
32. 240
33. 210
34. 126
35. 56
36. 1000

2.2 The Greatest Common Factor (GCF) & The Least Common Multiple (LCM)

In this section, Venn diagrams will be used to examine the similarities and differences in the prime factorization of numbers. Recall from Section 1.1 that a **Venn diagram** is a drawing, in which circular areas represent groups of items sharing common properties. The drawing consists of two or more circles, each representing a specific group.

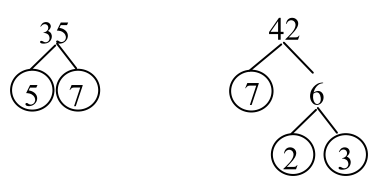
Each Venn diagram begins with a rectangle representing the universal set. The universal set will be the set of all real numbers. Then, each set in the problem is represented by a circle. The sets in this section will be the prime factorization of a number. Any values that belong to more than one set are **common factors** and will be placed in the section where the circles overlap.

THE GREATEST COMMON FACTOR (GCF)

The **Greatest Common Factor (GCF)** is the greatest factor that is the same in two or more numbers. (The largest number that can divide evenly into all the numbers.)

To find the greatest common factor, start by using a factor tree to find the prime factors of each number. Then create a Venn diagram to examine the factors. Place the common factors in the overlapping section of the circles. Place the remaining factors in the circle representing that set of factors. If there is more than one common factor, the greatest common factor is the product of all the common factors. If there are no common factors, the greatest common factor is 1 because 1 is a factor for all whole numbers.

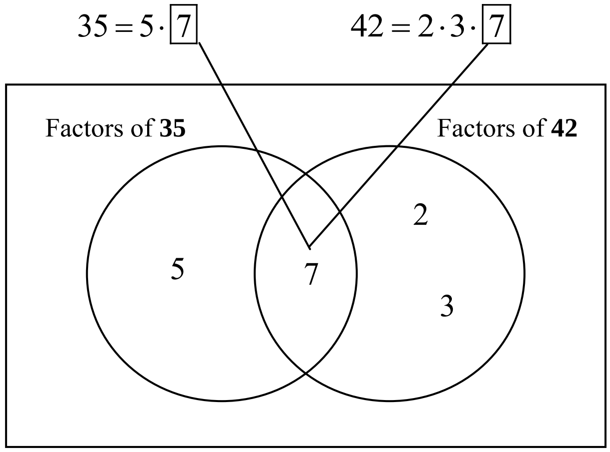
Example 1 **Find the greatest common factor of 35 and 42.**



Prime factor each number.

$35 = 5 \cdot \boxed{7}$ $\qquad$ $42 = 2 \cdot 3 \cdot \boxed{7}$

Write as a product of prime factors.

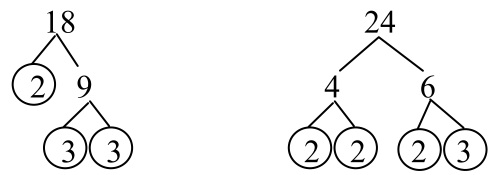


Use a Venn diagram to show the factors. The common factor, 7, is placed in the overlapping section of the circles. Place the remaining factors in the circle representing that set of factors.

$\boxed{\text{GCF} = 7}$

There is only one common factor, so this is the greatest common factor.

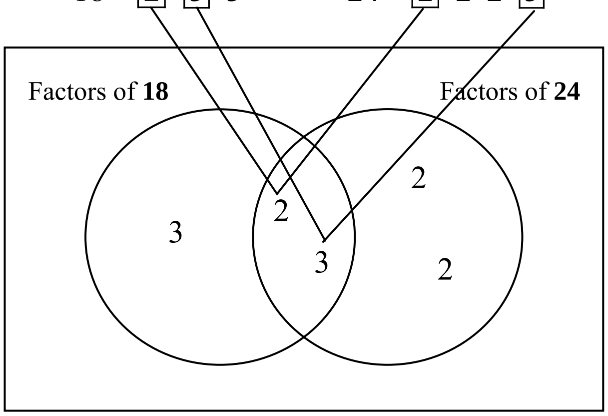
Example 2 **Find the greatest common factor of 18 and 24.**



Prime factor each number.

$18 = \boxed{2} \cdot \boxed{3} \cdot 3 \qquad 24 = \boxed{2} \cdot 2 \cdot 2 \cdot \boxed{3}$

Write as a product of prime factors.



Use a Venn diagram to show the factors. The common factors, 2 and 3, are placed in the overlapping section of the circles. Place the remaining factors in the circle representing that set of factors.

$\text{GCF} = 2 \cdot 3$

$\boxed{\text{GCF} = 6}$

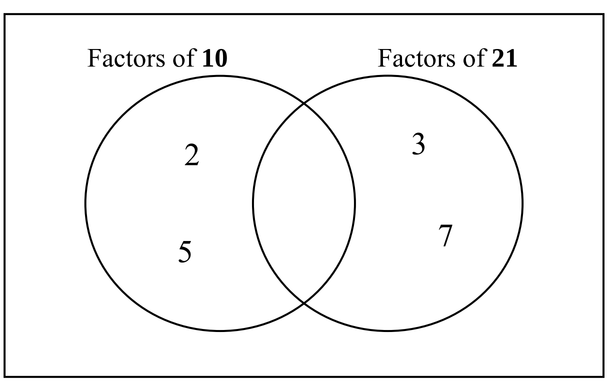
Multiply the common factors to get the greatest common factor.

Example 3 **Find the greatest common factor of 10 and 21.**

Prime factor each number.

$10 = 2 \cdot 5 \qquad 21 = 3 \cdot 7$

Write as a product of prime factors.



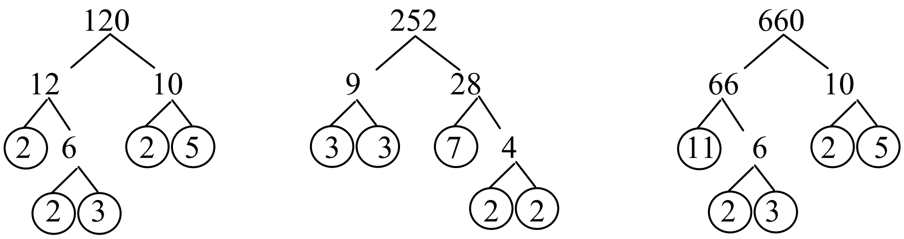
Use a Venn diagram to show the factors. In this example, there are no common factors to be placed in the overlapping section of the circles. Place the remaining factors in the circle representing that set of factors.

$\boxed{\text{GCF} = 1}$

There are no common factors, so the greatest common factor is 1.

Note: The GCF of two prime numbers is always 1. The greatest common factor of a prime number and a composite number will depend on whether the prime number is a factor of the composite number. If it is a factor then the greatest common factor is the prime number; if not, the greatest common factor is 1.

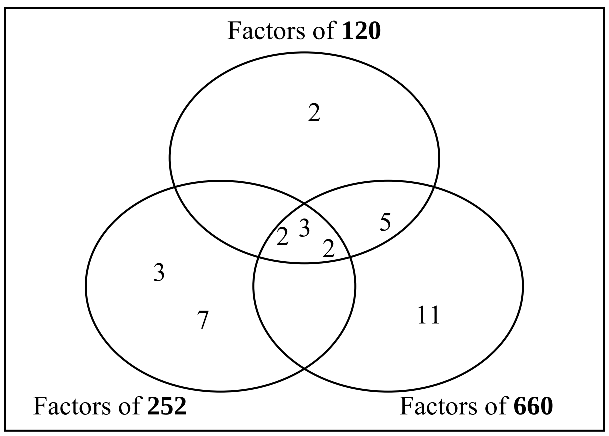
Example 4 **Find the greatest common factor of 120, 252, and 660.**



Prime factor each number.

$120 = \boxed{2\cdot 2}\cdot 2\cdot \boxed{3}\cdot ⑤$ $\quad 252 = \boxed{2\cdot 2}\cdot \boxed{3}\cdot 3\cdot 7$ $\quad 660 = \boxed{2\cdot 2}\cdot \boxed{3}\cdot ⑤\cdot 11$

Write as a product of prime factors.



Use a Venn diagram to show the factors. The common factors, 2, 2, and 3 are placed in the overlapping section of the circles. Place the remaining factors in the circle representing that set of factors.

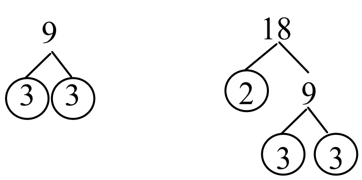
$$\text{GCF} = 2\cdot 2\cdot 3$$

$$\boxed{\text{GCF} = 12}$$

Multiply the common factors to get the greatest common factor.

The next three examples will demonstrate how to find the greatest common factor without the use of a Venn diagram.

Example 5 **Find the greatest common factor of 9 and 18.**



Prime factor each number.

$9 = \boxed{3}\cdot\boxed{3}$ $\quad 18 = 2\cdot\boxed{3}\cdot\boxed{3}$

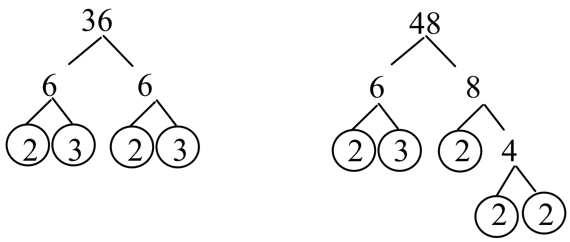
Write as a product of prime factors. Identify and write the common factors.

$$\text{GCF} = \boxed{3}\cdot\boxed{3}$$

$$\boxed{\text{GCF} = 9}$$

There are two factors in common. Multiply the common factors to get the greatest common factor.

Example 6 **Find the greatest common factor of 36 and 48.**



Prime factor each number.

$36 = \boxed{2} \cdot \boxed{2} \cdot \boxed{3} \cdot 3 \qquad 48 = \boxed{2} \cdot \boxed{2} \cdot 2 \cdot 2 \cdot \boxed{3}$

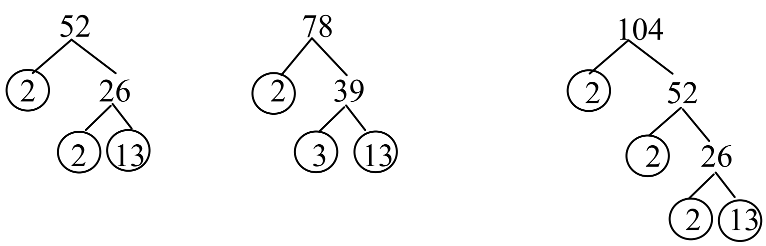
Write as a product of prime factors. Identify and write the common factors.

$\text{GCF} = \boxed{2} \cdot \boxed{2} \cdot \boxed{3}$

There are three factors in common. Multiply the common factors to get the greatest common factor.

$\boxed{\text{GCF} = 12}$

Example 7 **Find the greatest common factor of 52, 78, and 104.**



Prime factor each number.

$52 = \boxed{2} \cdot 2 \cdot \boxed{13} \qquad 78 = \boxed{2} \cdot 3 \cdot \boxed{13} \qquad 104 = \boxed{2} \cdot 2 \cdot 2 \cdot \boxed{13}$

Write as a product of prime factors. Identify and write the common factors.

$\text{GCF} = \boxed{2} \cdot \boxed{13}$

There are two factors in common. Multiply the common factors to get the greatest common factor.

$\boxed{\text{GCF} = 26}$

THE LEAST COMMON MULTIPLE (LCM)

A **multiple** of a number is the product of the number and any whole number. Some multiples of 2 and 3 are listed below.

$$\left.\begin{array}{l} 0\cdot 2=0 \\ 1\cdot 2=2 \\ 2\cdot 2=4 \\ 3\cdot 2=6 \\ 4\cdot 2=8 \\ 5\cdot 2=10 \\ 6\cdot 2=12 \\ \quad\vdots \end{array}\right\}\text{multiples of 2} \qquad \left.\begin{array}{l} 0\cdot 3=0 \\ 1\cdot 3=3 \\ 2\cdot 3=6 \\ 3\cdot 3=9 \\ 4\cdot 3=12 \\ 5\cdot 3=15 \\ 6\cdot 3=18 \\ \quad\vdots \end{array}\right\}\text{multiples of 3}$$

If you look closely at the multiples above you will notice that 0, 6, and 12 are multiples of both 2 and 3. These are **common multiples.** The smallest number (other than 0) that is a multiple of two or more whole numbers is the **Least Common Multiple (LCM)** of the numbers. The least common multiple of 2 and 3 is 6.

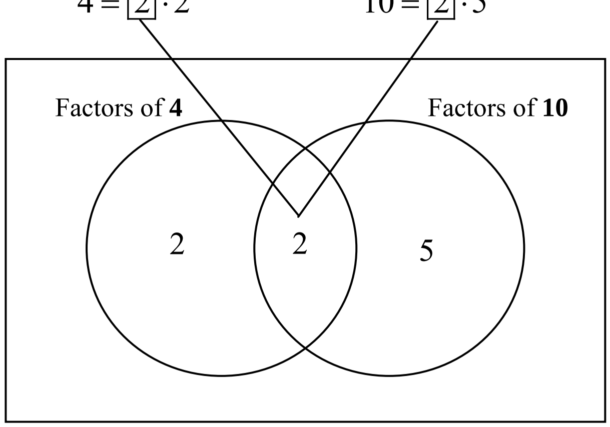
Venn diagrams, which were used as a tool in determining the GCF, can also be used as a tool to find the LCM. Start by prime factoring each number then, create a Venn diagram to examine the factors. The LCM will be the product of all the numbers in the Venn diagram.

Example 8 **Find the least common multiple of 4 and 10.**

Prime factor each number.

$4 = \boxed{2}\cdot 2 \qquad 10 = \boxed{2}\cdot 5$

Write as a product of prime factors.



Use a Venn diagram to show the factors. The common factor, 2, is placed in the overlapping section of the circles. Place the remaining factors in the circle representing that set of factors.

$\text{LCM} = 2\cdot 2\cdot 5$

$\boxed{\text{LCM} = 20}$

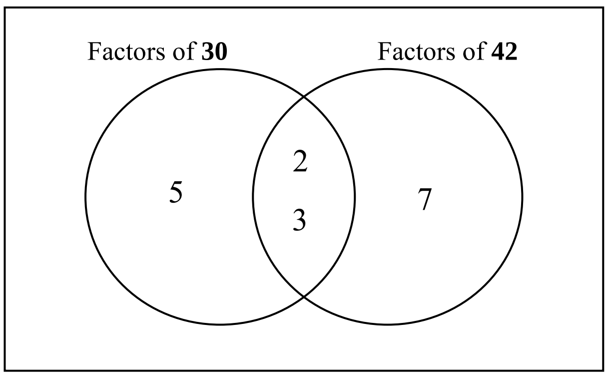
Multiply the common factor, 2, and all the remaining factors, 2 and 5, to get the least common multiple. (Find the product of all numbers in the Venn diagram.)

Example 9 Find the least common multiple of 30 and 42.

Prime factor each number.

$30 = 2 \cdot 3 \cdot 5$ $\qquad$ $42 = 2 \cdot 3 \cdot 7$

Write as a product of prime factors.



Use a Venn diagram to show the factors. The common factors, 2 and 3, are placed in the overlapping section of the circles. Place the remaining factors in the circle representing that set of factors.

$\text{LCM} = 2 \cdot 3 \cdot 5 \cdot 7$

$\boxed{\text{LCM} = 210}$

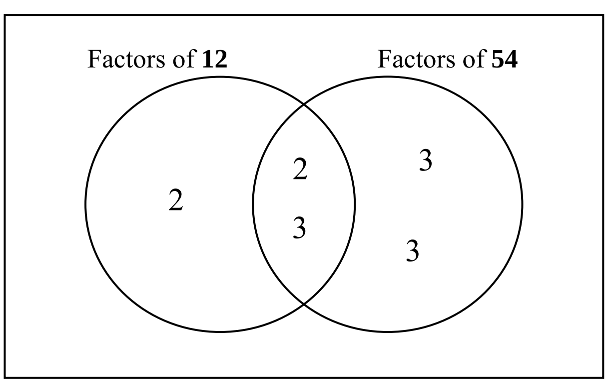
Multiply the common factors, 2 and 3, and all the remaining factors, 5 and 7, to get the least common multiple. (Find the product of all numbers in the Venn diagram.)

Example 10 Find the least common multiple of 12 and 54.

Prime factor each number.

$12 = 2 \cdot 2 \cdot 3$ $\qquad$ $54 = 2 \cdot 3 \cdot 3 \cdot 3$

Write as a product of prime factors.



Use a Venn diagram to show the factors. The common factors, 2 and 3, are placed in the overlapping section of the circles. Place the remaining factors in the circle representing that set of factors.

Multiply the common factors, 2 and 3, and all the remaining factors, 2, 3, and 3, to get the least common multiple. (Find the product of all numbers in the Venn diagram.)

$\text{LCM} = 2 \cdot 3 \cdot 2 \cdot 3 \cdot 3$

$\boxed{\text{LCM} = 108}$

The next two examples will demonstrate how to find the least common multiple without the use of a Venn diagram.

Example 11 **Find the least common multiple of 15 and 25.**

Prime factor each number.

$15 = 3 \cdot \boxed{5}$ $25 = 5 \cdot \boxed{5}$

Write as a product of prime factors. Identify and write the common factors.

$\text{LCM} = \boxed{5} \cdot 3 \cdot 5$

There is one factor in common. Multiply the common factor and all the remaining factors to get the least common multiple.

$\boxed{\text{LCM} = 75}$

Example 12 **Find the least common multiple of 8 and 18.**

Prime factor each number.

Write as a product of prime factors. Identify and write the common factors.

$8 = \boxed{2} \cdot 2 \cdot 2$ $18 = \boxed{2} \cdot 3 \cdot 3$

There is one factor in common. Multiply the common factor and all the remaining factors to get the least common multiple.

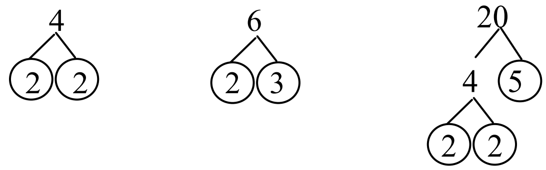
$\text{LCM} = \boxed{2} \cdot 2 \cdot 2 \cdot 3 \cdot 3$

$\boxed{\text{LCM} = 72}$

The least common multiple can be found with more than just two numbers. When finding the least common multiple with three or more numbers, first identify the factors that are in common in ***all*** of the numbers. Then look for the factors that are in common in ***any two*** of the numbers. Multiply the common factors with the left over factors that are not in common in any of the numbers to find the least common multiple.

The next two examples will demonstrate how to find the least common multiple of three numbers. The first example will use a Venn diagram, while the second example will not use a Venn diagram.

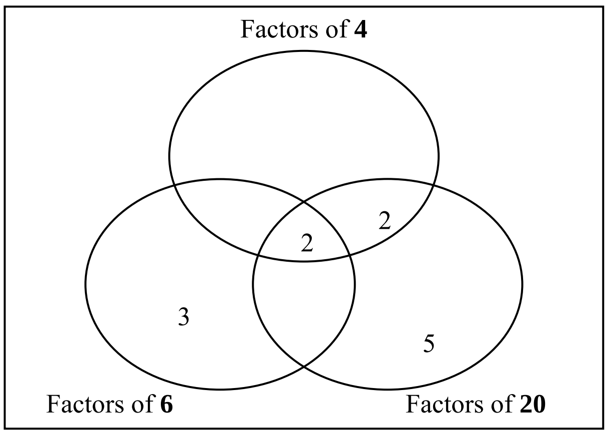
Example 13 **Find the least common multiple of 4, 6, and 20.**



Prime factor each number.

$4 = 2 \cdot 2$ $\quad 6 = 2 \cdot 3$ $\quad 20 = 2 \cdot 2 \cdot 5$

Write as a product of prime factors.



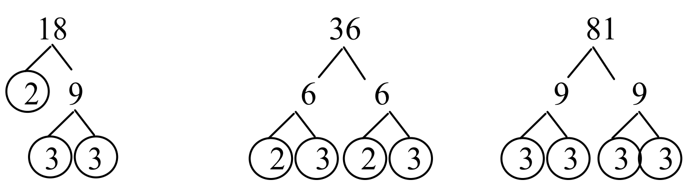
Use a Venn diagram to show the factors. The common factor found in all three numbers, 2, is placed in the overlapping section of the three circles. The second 2 that is in common with the numbers 4 and 20 is placed in the overlapping section of the circles representing the numbers 4 and 20. The remaining factors are placed in the circle representing that set of factors.

$$\text{LCM} = 2 \cdot 2 \cdot 3 \cdot 5$$

$$\boxed{\text{LCM} = 60}$$

Multiply the common factors and all the remaining factors to get the least common multiple. (Find the product of all numbers in the Venn diagram.)

Example 14 **Find the least common multiple of 18, 36, and 81.**



Write as a product of prime factors.

$18 = \Diamond 2 \cdot \boxed{3} \cdot \boxed{3}$ $\quad 36 = \Diamond 2 \cdot 2 \cdot \boxed{3} \cdot \boxed{3}$ $\quad 81 = 3 \cdot 3 \cdot \boxed{3} \cdot \boxed{3}$

Write as a product of prime factors. Identify and write the common factors from <u>all three terms.</u> Then identify and write any common factors from <u>two of the terms</u>. (The diamond around the two indicates that it is in common in two of the terms.)

$$\text{LCM} = \boxed{3} \cdot \boxed{3} \cdot \Diamond 2 \cdot 2 \cdot 3 \cdot 3$$

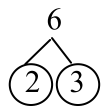
$$\boxed{\text{LCM} = 324}$$

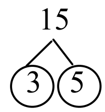
Multiply the common factor and all the remaining factors to get the least common multiple.

Note: To help you differentiate between the greatest common factor (GCF) and the least common multiple (LCM) you can think that both include the common factors but the **L**east common multiple also includes the **L**eft over factors. (Least common multiple and Leftovers both start with **L**.)

Example 15

Find the greatest common factor and the least common multiple of 6 and 15.

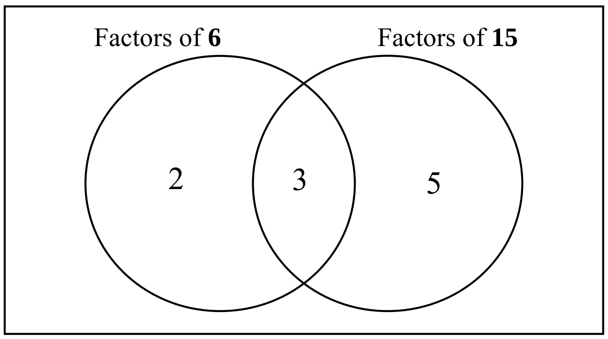




Prime factor each number.

$6 = 2 \cdot 3$ $\qquad$ $15 = 3 \cdot 5$

Write as a product of prime factors.



Use a Venn diagram to show the factors. Place the common factors in the overlapping section of the circles. Place the remaining factors in the circle representing that set of factors.

$\boxed{GCF = 3}$

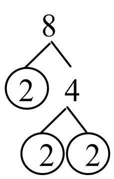
$LCM = 3 \cdot 2 \cdot 5$

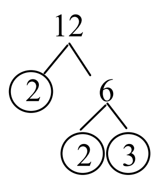
$\boxed{LCM = 30}$

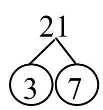
The GCF is the product of the numbers that are in common. The LCM is the product of the common factors and the factors that are not in common.

Example 16

Find the greatest common factor and the least common multiple of 8, 12, and 21.



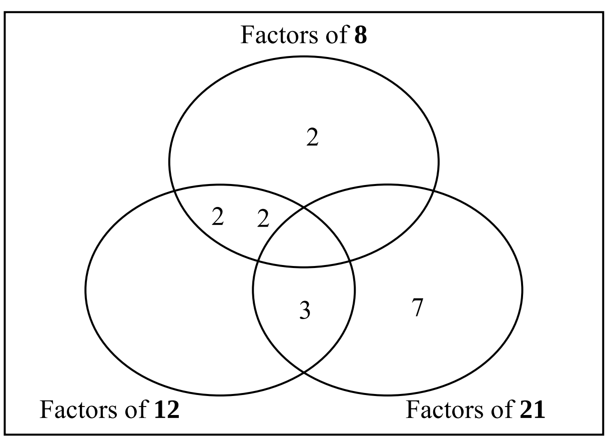




Prime factor each number.

$8 = 2 \cdot 2 \cdot 2$ $\qquad$ $12 = 2 \cdot 2 \cdot 3$ $\qquad$ $21 = 3 \cdot 7$

Write as a product of prime factors.



Use a Venn diagram to show the factors. Place the common factors in the overlapping section of the circles. Place the remaining factors in the circle representing that set of factors.

$LCM = 2 \cdot 2 \cdot 3 \cdot 2 \cdot 7$

$\boxed{GCF = 1}$ $\qquad$ $\boxed{LCM = 168}$

The GCF is the product of the numbers that are in common. The LCM is the product of the common factors and the factors that are not in common.

2.2 EXERCISES

In 1-4, use a Venn diagram to find the greatest common factor of the following sets of numbers.

1. 8 and 12
2. 24 and 40
3. 120 and 216
4. 15, 20, and 30

In 5-12, find the greatest common factor of the following sets of numbers.

5. 12 and 35
6. 18 and 42
7. 9 and 10
8. 12 and 48
9. 40, 50, and 60
10. 18, 30, and 42
11. 72, 108, and 180
12. 700, 420, and 1,120

In 13-16, use a Venn diagram to find the least common multiple of the following sets of numbers.

13. 3 and 4
14. 4 and 6
15. 4, 6, and 18
16. 3, 21, and 56

In 17-24, find the least common multiple of the following sets of numbers.

17. 25 and 80
18. 9 and 15
19. 12 and 20
20. 15 and 25
21. 70, 80, and 90
22. 10, 15, and 100
23. 7, 12, and 28
24. 11, 33, and 121

In 25-30, find the greatest common factor and the least common multiple of the following sets of numbers.

25. 12 and 18
26. 45 and 60
27. 27 and 72
28. 6, 8, and 24
29. 10, 25, and 30
30. 16, 64, and 72

2.3 Exponents

In mathematics, there are shorthand ways of writing things. **Exponents** are a shorthand way to write repeated multiplication. An exponent can also be referred to as a power of the base. The exponent is the number of times the base is used as a factor. For example, $3\cdot3\cdot3\cdot3$ can be written as 3^4. (3 multiplied by itself 4 times.) The number 3 is called the base and the number 4 is called the exponent. $3\cdot3\cdot3\cdot3$ is considered to be in *expanded form* while 3^4 is considered to be the *exponential form.*

Exponents are not only used with numbers. They can be used with the letters of mathematics, called **variables**. (Variables will be discussed in more detail in chapter 5) For example, $x\cdot x\cdot x\cdot x\cdot x$ can be written as x^5, where the variable, *x*, is the base and the number, 5, is the exponent. An important concept to understand when working with numbers and variables occurs when a number is placed before a variable as in 5*x*. The operation performed is multiplication. 5*x* is the same as 5 times *x,* which is not the same as x^5. (Recall from section 1.6, when two items are placed next to one another, it represents the operation of multiplication.)

How do exponents affect negative numbers? Look at the following examples of the exponents written in expanded form.

$$\begin{aligned}(-2)^4 &= (-2)(-2)(-2)(-2)\\ &= 16\end{aligned}$$

When the negative is in the parentheses, the negative is included in the repeated multiplication.

$$\begin{aligned}-2^4 &= -2\cdot2\cdot2\cdot2\\ &= -16\end{aligned}$$

When the negative is not in the parentheses, the negative is not included in the repeated multiplication. It is written once in the front.

Numbers and variables with no exponents showing are understood to have an exponent of 1. In the example, $3x^2$, the 3 has an exponent of 1 and the *x* has an exponent of 2, allowing us to write it in expanded form as $3\cdot x\cdot x$.

The most common error that occurs when working with examples such as the one above is to forget the fact that if an exponent is not showing it is understood to be a 1. This error causes some to attach the exponent on the variable to both the number and the variable. In our example, $3x^2$, this error is seen when the 3 is written twice and the variable is written twice. The proper way to write $3x^2$ in expanded form is $3\cdot x\cdot x$ not ~~$3\cdot3\cdot x\cdot x$~~. **Recall, numbers and variables with no exponent showing are understood to have an exponent of 1.**

Example 1 **Rewrite $4\cdot4\cdot4$ in exponential form.**

$4\cdot4\cdot4=\boxed{4^3}$ *4 multiplied by itself 3 times.*

Example 2

Rewrite $(-5)(-5)(-5)(-5)(-5)(-5)$ in exponential form.

$$(-5)(-5)(-5)(-5)(-5)(-5) = \boxed{(-5)^6}$$

(–5) multiplied by itself 6 times. The negative must be enclosed in the parentheses because it is included in the repeated multiplication.

Example 3

Rewrite $-3 \cdot 3 \cdot x \cdot x \cdot x \cdot x$ in exponential form.

$$-3 \cdot 3 \cdot x \cdot x \cdot x \cdot x = \boxed{-3^2 x^4}$$

*The negative is **not** enclosed in the parentheses because it is **not** included in the repeated multiplication. 3 multiplied by itself 2 times and x multiplied by itself 4 times.*

Example 4

Rewrite 11 in exponential form.

$$11 = \boxed{11^1}$$

A number and/or a variable written only once has an exponent of 1.

Example 5

Rewrite $(x+1)(x+1)(x+1)$ in exponential form.

$$(x+1)(x+1)(x+1) = \boxed{(x+1)^3}$$

(x + 1) multiplied by itself 3 times.

Example 6

Rewrite 6^2 in expanded form.

$$6^2 = \boxed{6 \cdot 6}$$

6 multiplied by itself 2 times.

Example 7

Rewrite $(-7)^4$ in expanded form.

$$(-7)^4 = \boxed{(-7)(-7)(-7)(-7)}$$

(–7) multiplied by itself 4 times. Notice the negative must be used in the repeated multiplication.

Example 8 **Rewrite $8a^3$ in expanded form.**

$8a^3 = \boxed{8 \cdot a \cdot a \cdot a}$

8 times "a" multiplied by itself 3 times.

Example 9 **Rewrite -2^3x^5 in expanded form.**

$-2^3x^5 = \boxed{-2 \cdot 2 \cdot 2 \cdot x \cdot x \cdot x \cdot x \cdot x}$

*The negative is **not** used in the repeated multiplication.*

Example 10 **Rewrite 9^1 in expanded form.**

$9^1 = \boxed{9}$

9 written 1 time.

Example 11 **Rewrite $(-2x)^3$ in expanded form.**

$(-2x)^3 = \boxed{(-2x)(-2x)(-2x)}$

-2x written 3 times.

When simplifying mathematics problems that involve numbers raised to exponents, we can find the value of the problem by multiplying. Observe the following examples.

Example 12 **Find the value of 4^3.**

$$4^3 = 4 \cdot 4 \cdot 4$$
$$= \boxed{64}$$

4 multiplied by itself 3 times.

Example 13 **Find the value of 12^1.**

$12^1 = \boxed{12}$

12 written 1 time.

Example 14 **Rewrite -3^4 in expanded form, then find its value.**

$$-3^4 = -3 \cdot 3 \cdot 3 \cdot 3$$
$$= \boxed{-81}$$

*The negative is **not** used in the repeated multiplication.*

There are several rules that apply to exponents but, before we can approach this subject we need to gain a stronger mathematics background. This topic will be addressed in more detail in Chapter 9. One rule that is important to memorize for exponents is the **Zero Power Rule**.

> **ZERO POWER RULE FOR EXPONENTS**
>
> For all real numbers *a*, where $a \neq 0$,
>
> $$a^0 = 1$$
>
> *(Any non-zero base raised to the power of 0 equals 1. This rule will be discussed in more detail in chapter 9. For now, learn it!)*

For example:

$$5^0 = 1 \qquad -\left(\frac{1}{4}\right)^0 = -1 \qquad (-2)^0 = 1 \qquad -8^0 = -1 \qquad 125^0 = 1$$

Example 15 **Find the value of 3^0.**

$3^0 = \boxed{1}$

A base raised to the power of 0 will always equal 1.

Example 16 **Find the value of $\left(-\frac{2}{3}\right)^0$.**

$\left(-\frac{2}{3}\right)^0 = \boxed{1}$

A base raised to the power of 0 will always equal 1. The negative is enclosed in the parentheses so it is also raised to the zero power.

Example 17 **Find the value of $-1,432^0$.**

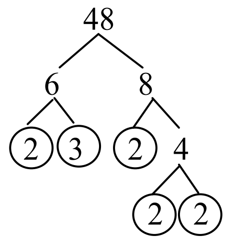
$-1,432^0 = \boxed{-1}$

*A base raised to the power of 0 will always equal 1. The negative is **not** enclosed in the parentheses so it is **not** raised to the zero power. The answer is negative.*

In Section 2.2, a factor tree was used to find the prime factorization of a number. The prime factorization can be written in exponential form. The following examples illustrate this concept.

Example 18

Find the prime factorization of 48. Express your answer in exponential form.



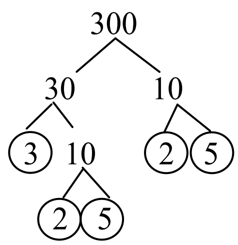
The prime factorization of 48 is:

$$48 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$$

$$\boxed{48 = 2^4 \cdot 3}$$

Example 19

Find the prime factorization of 300. Express your answer in exponential form.



The prime factorization of 300 is:

$$300 = 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5$$

$$\boxed{300 = 2^2 \cdot 3 \cdot 5^2}$$

2.3 EXERCISES

In 1-6, rewrite the following with an exponent.

1. $2 \cdot 2 \cdot 2 \cdot a \cdot a$

2. $(-6)(-6)(-6)$

3. $-4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$

4. $3 \cdot x \cdot x \cdot y \cdot z \cdot z \cdot z$

5. $(x)(x)$

6. $-7 \cdot 7 \cdot 7 \cdot 7 \cdot x \cdot x \cdot y \cdot y \cdot y \cdot z$

In 7-12, rewrite the following in expanded form.

8. p^6

9. $(-5)^4$

10. -5^4

11. $-9x^3$

12. $7x^2y^5$

13. $8a^2b^0c^3$

In 13-18, find the value of the following.

13. -2^6

14. 1248^0

15. 11^2

16. $-\left(\frac{7}{8}\right)^0$

17. $(-3)^4$

18. $(-25)^0$

In 19-22, find the prime factorization value of the following numbers. Express your answer in exponential form.

19. 60

20. 98

21. 495

22. 2,520

In 23-26, read and respond to the following exercises.

23. Explain why $(4 \cdot 3)^2$ is not the same as $4 \cdot 3^2$.

24. Explain why -5^2 is not the same as $(-5)^2$.

25. Complete the following sentence. Exponents are the shorthand way to _____________.

26. Complete the following sentence. A base raised to the power of zero is ____________.

In 27-28, solve the following application problems.

27. In 1990, Sanpete County ranked 1st in the United States in total population of turkeys with more than 9^8 turkeys. What was the approximate turkey population of Sanpete County?

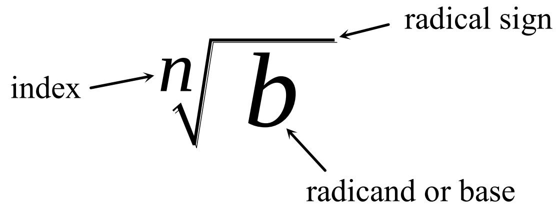
28. A new fast food restaurant opened in a small town in Southern Utah. During its first week of business the owner projected the number of customers to be served would be 15^3. At the end of the first week the owner calculated that they actually served 8^4 customers. How many customers were served above the owners projected amount?

2.4 Roots

ROOTS OR RADICALS

Finding a root of a number is the opposite operation of raising a number to a power or exponent. To learn the basics we will deal with roots of positive numbers only.

The following diagram illustrates the notation used for roots.



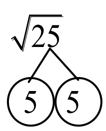
radical sign $\left(\sqrt{\ }\right)$**:** indicates that a root is to be found.

index (*n*): tells which root is to be found.

radicand or base (*b*): the number under the radical.

Note: $\sqrt{\ } = \sqrt[2]{\ }$ (If there is no index, it is the same as having an index of 2. This type of a root is known as a square root.)

To find a root, start by prime factoring the base. Look for groups of the same number. (The size of your group will depend on the index in your problem.) Once we have located a group of the same number, place that number on the outside of the radical. Study the following examples.

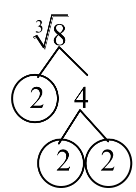


There is no index, so we know we are looking for a square root or groups of two. There are two 5's.

$$\sqrt{25} = \sqrt{\boxed{5 \cdot 5}}$$

The square root of 25 is **5**.

$$\sqrt{25} = \boxed{5}$$

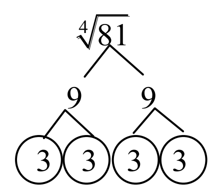


The index is a three. We are looking for groups of three. There are three 2's.

$$\sqrt[3]{8} = \sqrt[3]{\boxed{2 \cdot 2 \cdot 2}}$$

The cube root of 8 is **2**.

$$\sqrt[3]{8} = \boxed{2}$$



The index is four. We are looking for groups of four. There are four 3's.

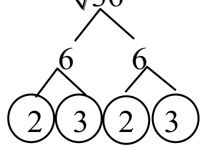
$$\sqrt[4]{81} = \sqrt[4]{\boxed{3 \cdot 3 \cdot 3 \cdot 3}}$$

The fourth root of 81 is **3**.

$$\sqrt[4]{81} = \boxed{3}$$

Example 1 **Find the indicated root.** $\sqrt{36}$

$$\sqrt{36} = \sqrt{\boxed{2\cdot 2}\cdot\boxed{3\cdot 3}} = 2\cdot 3$$

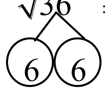


We are looking for groups of two. There are two 2's and two 3's. The square root will be the product of 2 and 3, which is 6.

$$\sqrt{36} = \boxed{6}$$

This problem can be done much quicker by identifying the two 6's.

$$\sqrt{36} = \sqrt{\boxed{6\cdot 6}}$$

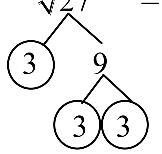


Identify a group of two.

$$\sqrt{36} = \boxed{6}$$

Example 2 **Find the indicated root.** $\sqrt[3]{27}$

$$\sqrt[3]{27} = \sqrt[3]{\boxed{3\cdot 3\cdot 3}}$$

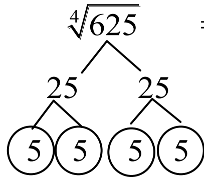


We are looking for groups of three. There are three 3's. The cube root of 27 is 3.

$$\sqrt[3]{27} = \boxed{3}$$

Example 3 **Find the indicated root.** $\sqrt[4]{625}$

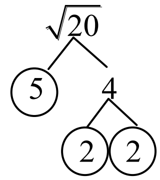
$$\sqrt[4]{625} = \sqrt[4]{\boxed{5\cdot 5\cdot 5\cdot 5}}$$



We are looking for groups of four. There are four 5's. The fourth root of 625 is 5

$$\sqrt[4]{625} = \boxed{5}$$

What happens when we don't have equal groups? Study the following example.



There is no index, so we are looking for groups of two. There is one group of 2's, but we have a 5 left over that is not in a group. When this happens, whatever does not form a group must stay under the radical sign.

$$\sqrt{20} \quad = \quad \sqrt{\boxed{2\cdot 2}\cdot 5} \quad = \quad 2\sqrt{5}$$

THE RADICAL HOUSE

There is a fun story to help you understand the concept of radicals. The story is called "The *Radical House*". The radical sign $\left(\sqrt{\ }\right)$ can be thought of as a *radical* college house. The prime factors of the radicand will represent the students in that house. Most college students love a good party, especially those in the *radical* house.

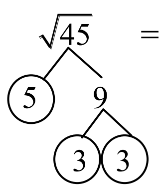
The students in the *radical* house have been invited to an outdoor costume party. In order to leave the *radical* house and attend the party, they must come with a group dressed in the same costume. The number of people in the group depends on the index of the radical. (With a square root, there must be two people in the group, a cube root must have three people in the group, etc.)

If the students can find a group of matching costumes, they can go to the party. When leaving the house and attending the party, they stand as one group. If there are any students without a group, they must stay in the *radical* house.

The next three examples will relate simplifying radicals to The Radical House story.

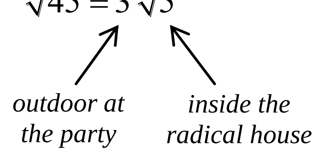
Example 4 **Find the indicated root.** $\sqrt{45}$

$$\sqrt{45} \quad = \quad \sqrt{\boxed{3\cdot 3}\cdot 5}$$



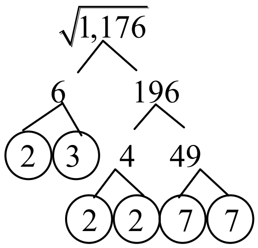
The students in the radical house have been invited to a party that requires them to come in groups of two. (Prime factor 45.) There are two 3's so the 3's get to go to the outdoor party. The 5 must stay in the radical house because he does not have a group to attend the party.

$$\sqrt{45} = 3\sqrt{5}$$



Write the 3's as a group outside the radical house and write the 5 inside the radical house.

Example 5 **Find the indicated root.** $\sqrt{1{,}176}$



$$\sqrt{1{,}176} = \sqrt{\boxed{2\cdot 2}\cdot 2\cdot 3\cdot \boxed{7\cdot 7}}$$

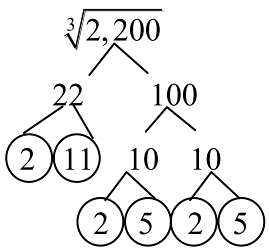
The students in the radical house have been invited to a party that requires them to come in groups of two. (Prime factor 1,176.) There are two 2's and two 7's. They get to go to the outdoor party. One 2 and the 3 must stay in the radical house because they do not have a group to attend the party.

$$\sqrt{1{,}176} = 2\cdot 7\sqrt{2\cdot 3} = 14\sqrt{6}$$

outdoor at the party ↗ ↖ *inside the radical house*

Write the 2's and the 7's as a group outside the radical house. Write the remaining 2 and the 3 inside the radical house. Simplify by multiplying.

Example 6 **Find the indicated root.** $\sqrt[3]{2{,}200}$



$$\sqrt[3]{2{,}200} = \sqrt{\boxed{2\cdot 2\cdot 2}\cdot 5\cdot 5\cdot 11}$$

The students in the radical house have been invited to a party that requires them to come in groups of three. (Prime factor 2,200.) There are three 2's, two 5's and one 11. They 2's get to go to the outdoor party. The 5's and the 11 must stay in the radical house because they do not have a group to attend the party.

$$\sqrt[3]{2{,}200} = 2\sqrt[3]{5\cdot 5\cdot 11} = 2\sqrt[3]{275}$$

outdoor at the party ↗ ↖ *inside the radical house*

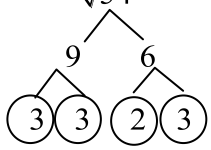
Write the 2's as a group outside the radical house. Write the 5's and the 11 inside the radical house. Simplify by multiplying.

The previous three examples have related simplifying radicals to the radical house. The remainder of the examples in this section will not specifically refer to the radical house. Simplify the radicals in the same manner. Prime factor the radicand and look for groups of the same number. The groups go on the outside of the radical, the factors without groups remain under the radical. If there is more than one factor either inside or outside the radical, multiply to simplify the answer.

Example 7

Find the indicated root. $\sqrt[3]{54}$

$$\sqrt[3]{54} = \sqrt[3]{2\cdot\boxed{3\cdot3\cdot3}}$$



The index is 3, so we are looking for groups of three. There are three 3's and one two.

Place a 3 outside the radical for the complete group. The 2 must stay under the radical because it does not form a group.

$$\sqrt[3]{54} = \boxed{3\sqrt[3]{2}}$$

Example 8

Find the indicated root. $\sqrt{17}$

$$\sqrt{17} = \boxed{\sqrt{17}}$$

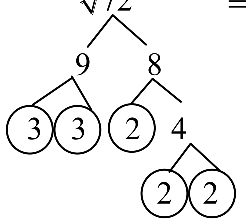
17 is a prime number. There are no groups.

Example 9

Find the indicated root. $\sqrt{72}$

$$\sqrt{72} = \sqrt{\boxed{2\cdot2}\cdot2\cdot\boxed{3\cdot3}}$$

Look for groups of two. There are two 3's and three 2's.



$$\sqrt{72} = 2\cdot3\sqrt{2} = \boxed{6\sqrt{2}}$$

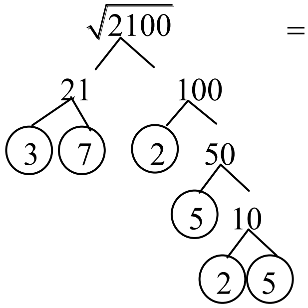
Place one 2 and one 3 on the outside. Leave one 2 inside the radical.

Example 10

Find the indicated root. $\sqrt{2100}$

$$\sqrt{2100} = \sqrt{\boxed{2\cdot2}\cdot3\cdot\boxed{5\cdot5}\cdot7}$$

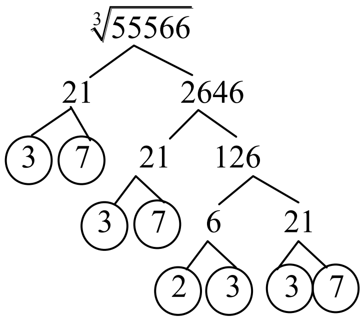
Look for groups of two. There are two 2's and two 5's.



Place one 2 and one 5 on the outside. Leave the 3 and the 7 inside the radical. Multiply.

$$\sqrt{2100} = 5\cdot2\sqrt{3\cdot7} = \boxed{10\sqrt{21}}$$

Example 11 **Find the indicated root.** $\sqrt[3]{55566}$



Look for groups of three.

$$\sqrt[3]{2\cdot\boxed{3\cdot3\cdot3}\cdot3\cdot\boxed{7\cdot7\cdot7}}$$

There are three 3's and three 7's.

$$\sqrt[3]{55566} = 3\cdot7\sqrt[3]{2\cdot3} = \boxed{21\sqrt[3]{6}}$$

Place one 3 and one 7 on the outside. Leave the 2 and one 3 inside the radical. Multiply.

The following table of perfect squares and perfect cubes may help you with roots. This table is also located in Appendix A at the back of your book.

SQUARES	SQUARE ROOTS	CUBES
$1^2=1$	$\sqrt{1}=1$	$1^3=1$
$2^2=4$	$\sqrt{4}=2$	$2^3=8$
$3^2=9$	$\sqrt{9}=3$	$3^3=27$
$4^2=16$	$\sqrt{16}=4$	$4^3=64$
$5^2=25$	$\sqrt{25}=5$	$5^3=125$
$6^2=36$	$\sqrt{36}=6$	$6^3=216$
$7^2=49$	$\sqrt{49}=7$	$7^3=343$
$8^2=64$	$\sqrt{64}=8$	$8^3=512$
$9^2=81$	$\sqrt{81}=9$	$9^3=729$
$10^2=100$	$\sqrt{100}=10$	$10^3=1000$
$11^2=121$	$\sqrt{121}=11$	$11^3=1331$
$12^2=144$	$\sqrt{144}=12$	$12^3=1728$

2.4 EXERCISES

In 1-20, find the indicated root.

1. $\sqrt{49}$
2. $\sqrt{124}$
3. $\sqrt[3]{125}$
4. $\sqrt[4]{162}$
5. $\sqrt[3]{81}$
6. $\sqrt[3]{64}$
7. $\sqrt{144}$
8. $\sqrt[3]{216}$
9. $\sqrt{625}$
10. $\sqrt[3]{343}$
11. $\sqrt{99}$
12. $\sqrt[5]{300,000}$
13. $\sqrt[3]{40}$
14. $\sqrt{252}$
15. $\sqrt[3]{512}$
16. $\sqrt{52}$
17. $\sqrt{19}$
18. $\sqrt{72}$
19. $\sqrt[4]{50,625}$
20. $\sqrt[3]{1,000,000}$

In 21-22, read and respond to the following exercises.

21. What is the relationship between squared numbers and square roots?

22. What is the index of a root and why is it so important?

2.5 Order of Operations & Evaluating Algebraic Expressions

ORDER OF OPERATIONS

We have discussed the mathematical operations of addition, subtraction, multiplication, division, evaluating exponents, and finding roots. The majority of the problems to this point involved only one of these operations. What happens when we have a problem that contains more than one of these operations? Which operation do we perform first? One might have the tendency to perform the operations in order from left to right, because that is how we read. This is not necessarily the case. Not all languages are read from left to right, some are read top to bottom. Mathematics is a universal language. Because of this, the **Order of Operations** were established. The Order of Operations guarantee that the steps will be done in the same order regardless of the language you speak or the country you are in. The following is a table that outlines the steps to the Order of Operations.

ORDER OF OPERATIONS

1. Simplify within **parentheses** ( ) and other grouping symbols, such as **brackets** [ ], **braces** { }, or the **fraction bar** —. *(When more than one pair of grouping symbols occur within a problem, work the innermost set of grouping symbols first.)*
2. Evaluate **exponents** and/or **roots**.
3. **Multiply** and/or **divide** in order from left to right.
4. **Add** and/or **subtract** in order from left to right.

When following the steps in the Order of Operations, we must complete each step, if applicable, before going on to the next. In steps 3 and 4, multiplication does not take precedence over division and addition does not take precedence over subtraction. Perform these operations **in order** from left to right.

Example 1 **Simplify** $5+2\cdot 8$

$5+\underbrace{2\cdot 8}$ — *There are no parentheses, exponents or roots, so we will begin with multiplication.*

$\underbrace{5+16}$ — *Add.*

$\boxed{21}$

Example 2

Simplify. $24-(5-2)\cdot 6$

$24-(\underbrace{5-2})\cdot 6$ — *Start by simplifying within the parentheses.*

$24-\underbrace{3\cdot 6}$ — *Multiply.*

$\underbrace{24-18}$ — *Subtract.*

$\boxed{6}$

Example 3

Simplify. $(9-8)^3+3\cdot 2^4$

$(\underbrace{9-8})^3+3\cdot 2^4$ — *Start by simplifying within the parentheses.*

$(1)^3+3\cdot 2^4$ — *Evaluate the exponents.*

$1+\underbrace{3\cdot 16}$ — *Multiply.*

$\underbrace{1+48}$ — *Add.*

$\boxed{49}$

Example 4

Simplify. $75\div 5\cdot 3-30+3$

$\underbrace{75\div 5}\cdot 3-30+3$

There are no parentheses, exponents, or roots. Begin with multiplication and division. Perform these operations in order from left to right. Because the operation of division is on the left, do this first. Then multiply.

$\underbrace{15\cdot 3}-30+3$

$\underbrace{45-30}+3$ — *Add or subtract in order from left to right.*

$\underbrace{15+3}$

$\boxed{18}$

Example 5

Simplify. $5 \cdot 9 - (4+8) \div 2 - (7 \times 4)$

$5 \cdot 9 - \underbrace{(4+8)} \div 2 - \underbrace{(7 \times 4)}$ *Start by simplifying within the parentheses.*

$\underbrace{5 \cdot 9} - 12 \div 2 - 28$ *Multiply.*

$45 - \underbrace{12 \div 2} - 28$ *Divide.*

$\underbrace{45 - 6} - 28$ *Subtract. Work left to right.*

$\underbrace{39 - 28}$ *Subtract.*

$\boxed{11}$

Example 6

Simplify. $\dfrac{60 - 5^2 + 1}{-3(1+1)}$

$\dfrac{60 - 5^2 + 1}{-3(1+1)}$ *The fraction bar serves as a grouping symbol. Completely simplify the top and completely simplify the bottom before we divide. Start by simplifying within the parentheses on the bottom.*

$\dfrac{60 - 5^2 + 1}{-3(2)}$ *Evaluate the exponent.*

$\dfrac{60 - 25 + 1}{-3(2)}$ *Multiply.*

$\dfrac{60 - 25 + 1}{-6}$ *Add or subtract in order from left to right.*

$\dfrac{35 + 1}{-6}$

$\dfrac{36}{-6}$ *Divide.*

$\boxed{-6}$

Example 7

Simplify. $7\left[21-8\left(\sqrt{25}-3\right)\right]$

$7\left[21-8\left(\sqrt{25}-3\right)\right]$	*Start by simplifying within the innermost parentheses. Find the value of the root.*
$7\left[21-8\left(5-3\right)\right]$	*Subtract within the innermost parentheses.*
$7\left[21-8\left(2\right)\right]$	*Multiply.*
$7\left[21-16\right]$	*Subtract.*
$7\left[5\right]$	*Multiply.*
$\boxed{35}$	

Example 8

Simplify. $1+5\left\{3-2\left[4^2-3\left(2+1\right)\right]\right\}$

$1+5\left\{3-2\left[4^2-3\left(2+1\right)\right]\right\}$	*Start by simplifying within the innermost parentheses. Add.*
$1+5\left\{3-2\left[4^2-3\left(3\right)\right]\right\}$	*Simplify within the brackets. Evaluate the exponent.*
$1+5\left\{3-2\left[16-3\left(3\right)\right]\right\}$	*Multiply within the brackets.*
$1+5\left\{3-2\left[16-9\right]\right\}$	*Subtract within the brackets.*
$1+5\left\{3-2\left[7\right]\right\}$	*Multiply within the braces.*
$1+5\left\{3-14\right\}$	*Subtract within the braces.*
$1+5\left\{-11\right\}$	*Multiply.*
$1+-55$	*Add.*
$\boxed{-54}$	

AVERAGE

A common application problem for the Order of Operation is finding **average**. To find the average, find the sum of your values, then divide that sum by the total number of values. For example, Beth's test scores are 97%, 81%, 91%, and 71%. Find her average test score.

$$\frac{97+81+91+71}{4}=\frac{340}{4}=\boxed{85\%}$$

Find the sum of all the values. Divide that sum by the number of values, in this case, 4.

Example 9

Find the average of 1, 9, –20, and –6.

$$\frac{1+9+(-20)+(-6)}{4}$$

Find the sum of all the values and divide that sum by the number of values.

$$\frac{-16}{4}$$

Divide.

$$\boxed{-4}$$

Example 10

Find the average of 7, 9, 0, 15, –3, –8, and –6.

$$\frac{7+9+0+15+(-3)+(-8)+(-6)}{7}$$

Find the sum of all the values and divide that sum by the number of values.

$$\frac{14}{7}$$

Divide.

$$\boxed{2}$$

Example 11

Brian, the center for the Badgers basketball team, scored 25 points, 15 points, 10 points, 23 points, and 45 points in the last five basketball games. Find his average points per game.

$$\frac{25+15+10+23+45}{5}$$

Find the sum of all the values and divide that sum by the number of values.

$$\frac{118}{5}$$

Divide.

$$23.6$$

$$\boxed{\text{Brian averaged 23.6 points per game.}}$$

EVALUATING ALGEBRAIC EXPRESSIONS

We will now apply the Order of Operations when simplifying algebraic expressions.

An **algebraic expression** is any single variable or number or any grouping of variables and numbers without an equal sign. Examples of algebraic expressions include:

$$5x, \qquad -2y+7, \qquad 4xz^2-6y+8, \qquad \frac{4a+2b^3}{5c}$$

To evaluate an algebraic expression, replace the variable(s) with their given numerical value(s). When doing this, place parentheses ( ) around the number. (Recall from Section 1.4, that a number enclosed in parentheses means the number holds its value.) Follow the Order of Operations to simplify the expression. Finding the value of the expression is also called **evaluating the expression** for the variable.

Example 12 **Evaluate $x + 3$ if $x = 4$.**

$x+3$	*Replace x with (4).*
$\downarrow$	
$(4)+3$	*Add.*
$\boxed{7}$	

Example 13 **Evaluate $2(r - 5)$ if $r = 3$.**

$2(r-5)$	*Replace r with (3).*
$\downarrow$	
$2((3)-5)$	*Simplify within the parentheses by subtracting.*
$2(-2)$	*Multiply.*
$\boxed{-4}$	

Example 14 **Evaluate $2(x + y)$ if $x = 4$ and $y = -1$.**

$2(x+y)$	*Replace x with (4) and y with (–1).*
$\downarrow \quad \downarrow$	
$2((4)+(-1))$	*Simplify within the parentheses by adding.*
$2(3)$	*Multiply.*
$\boxed{6}$	

Example 15

Evaluate $\frac{b+6}{a}$ if a = 2 and b = 8.

$\frac{b+6}{a}$ — *Replace a with (2) and b with (8).*

$\frac{(8)+6}{(2)}$ — *The fraction bar serves as a grouping symbol. Completely simplify the top and completely simplify the bottom before dividing. Start by adding on the top.*

$\frac{14}{2}$ — *Divide.*

$\boxed{7}$

Example 16

Evaluate $\sqrt{r}-3s^2$ if r = 16 and s = –5.

$\sqrt{r}-3s^2$ — *Replace r with (16) and x with (–5).*

Evaluate the root and exponent.

$\sqrt{(16)}-3(-5)^2$ — *Multiply.*

$4-3(25)$ — *Subtract.*

$4-75$

$\boxed{-71}$

Example 17

Evaluate $\frac{x^2+z-3}{y}$ if x = 5, y = 2 and z = 4.

$\frac{x^2+z-3}{y}$ — *Replace x with (5), y with (2), and z with (4).*

$\frac{(5)^2+(4)-3}{(2)}$ — *The fraction bar serves as a grouping symbol. Completely simplify the top and completely simplify the bottom before dividing. Start by evaluating the exponent.*

$\frac{25+4-3}{2}$ — *Add or subtract in order from left to right.*

$\frac{26}{2}$ — *Divide.*

$\boxed{13}$

In application problems you may find it necessary to first write an algebraic expression before you can evaluate the expression. Recall from Chapter 1 the words that represent the different mathematical operations.

Example 18

Write an algebraic expression for each of the following. Use *x* to represent "*a number*". Evaluate the expression when *x* = 2.

a) Twice a number

Algebraic Expression: $2\cdot \quad x$

~~**Twice**~~ ~~a number~~

$\boxed{2x}$

Evaluate when $x = 2$.

$2x$

$2(2)$

$\boxed{4}$

b) Negative ten minus a number

Algebraic Expression: $-10 \quad - \quad x$

~~Negative ten~~ ~~**minus**~~ ~~a number~~

$\boxed{-10-x}$

Evaluate when $x = 2$.

$-10-x$

$-10-(2)$

$\boxed{-12}$

c) Seven more than a number *(the word **than** will reverse the order)*

Algebraic Expression: $7 \quad + \quad x$

~~Seven~~ ~~**more**~~ (**than**) ~~a number~~

$\boxed{x+7}$

Evaluate when $x = 2$.

$x+7$

$(2)+7$

$\boxed{9}$

d) The product of a number and negative three

Algebraic Expression: $\cdot$ x -3

The ~~**product**~~ of ~~a number~~ and ~~negative three~~

$$x \cdot (-3)$$

$$\boxed{-3x}$$

Evaluate when $x = 2$.

$$-3x$$

$$-3(2)$$

$$\boxed{-6}$$

e) Eight times a number divided by four

Algebraic Expression: 8 $\cdot$ x $\div$ 4

~~Eight~~ ~~**times**~~ ~~a number~~ ~~**divided by**~~ ~~four~~

$$8 \cdot x \div 4$$

$$\boxed{8x \div 4}$$

Evaluate when $x = 2$.

$$8x \div 4$$

$$8(2) \div 4$$

$$16 \div 4$$

$$\boxed{4}$$

2.5 EXERCISES

In 1-20, use the order of operations to simplify the following expressions.

1. $5+2\cdot 3$

2. $24\div 3-10$

3. $72\div 4\cdot 2$

4. $8-3\cdot 5$

5. $4\cdot 2+9\cdot 3$

6. $-4+8\div 2$

7. $12+\dfrac{18}{(-3)}$

8. $(4+5)\div 3$

9. $\dfrac{18+6}{4-2^4}$

10. $\sqrt{81}+5-3^2$

11. $6^2\cdot(10-8)$

12. $\dfrac{-43-17}{3^2-4}$

13. $(5^2+5)\div 5$

14. $5\div 0+18$

15. $3^4-[35-(12-6)]$

16. $\dfrac{5(12-7)-4}{5^2-2^3-10}$

17. $(7\cdot 5)+[9\div(3\div 3)]$

18. $5+4[2-3(12\div 3+1)]$

19. $4+\{2-5[3^2+4(\sqrt{16}-3)]\}$

20. $29-\{5+3[8\cdot(\sqrt{100}-8)]-50\}$

In 21-25, find the average.

21. Find the average of 12, 7, 0, 8, and 23.

22. Find the average of –5, –10, 18, and –27.

23. Tommy's allowance for the last 6 weeks was $10, $12, $15, $10, $8, and $11. What was his average weekly allowance?

24. Cali's test scores in her English class were 84%, 73%, 98% and 90%. What was her average test score.

25. The morning temperatures for the last week were 7°, 9°, 4°, 0°, –3°, 5°, and –1°. What was the average morning temperature?

In 26-33, evaluate the following expressions for x = 5, y = –2, and z = 3.

26. $4+5x$

27. $2x+y$

28. $-3x-2y+z$

29. $2xy^2-5$

30. $\frac{3x-z}{y}$

31. $(2x-y)^2$

32. $xy(2x-3y+z)$

33. $\frac{y^3+z^2+18}{x}$

In 34-41, write an algebraic expression for the each of the following. Use x, to represent "a number". Evaluate the expression when x = 2.

34. The sum of three and a number.

35. The difference between a number and twelve.

36. The quotient of six and a number.

37. The product of a number and negative seven.

38. Fifteen less than a number.

39. Twenty-four decreased by twice a number.

40. The square of a number increased by negative seven.

41. The difference of three times a number and four.

CHAPTER 2 REVIEW EXERCISES

In 1-4, use a factor tree to write the prime factorization of each number.

1. 30
2. 79
3. 720
4. 1188

In 5-10, find the greatest common factor of the following.

5. 6 and 15
6. 18 and 35
7. -21 and 14
8. 12, -6, and 3
9. 27, 36, and 54
10. -18, 9, and 36

In 11-14, find the least common multiple of the following.

11. 5 and 30
12. 12 and 40
13. 15, 45, and 100
14. 6, 9, and 27

In 15-18, find the greatest common factor and the least common multiple of the following.

15. 16 and 64
16. 10 and 25
17. 8 and 15
18. 9, 12, and 18

In 19-22, rewrite the following with an exponent.

19. $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$
20. $(-2)(-2)(-2)x \cdot x \cdot x \cdot x$
21. $-1 \cdot 1 \cdot 1$
22. $5 \cdot 5 \cdot 5 \cdot a \cdot a \cdot a \cdot a \cdot b \cdot b \cdot c$

In 23-26, rewrite the following in expanded form.

23. x^3
24. -5^4
25. $2^3 x^2 y^3$
26. $-10x^4 yz^2$

In 27-30, find the value of the following.

27. 2^8

28. $(-4)^2$

29. -4^2

30. 211^0

In 31-34, find the indicated root.

31. $\sqrt{100}$

32. $\sqrt[3]{1080}$

33. $\sqrt[4]{625}$

34. $\sqrt[5]{320}$

In 35-40, use the order of operations to simplify the following expressions.

35. $6 \div 3 + 5^2$

36. $2^5 \div 4 \cdot 2 + \sqrt{9}$

37. $2 + 6\left[4^2 - (2-4)\right]$

38. $\dfrac{4(3+2)+4}{-4-2}$

39. $7\left[3-5(4-6)\right]$

40. $7^2 - \left\{18 - \left[40 \div (4 \cdot 2) + 2\right] + 5^2\right\}$

In 41-42, find the average.

41. Find the average of 12, 0, –1, –8, 10, 3, –7, and 15.

42. Ben earned 75%, 84% and 93% on his Biology exams. What is his average test score?

In 43-46, evaluate the following expressions for x = 2, y = –3, and z = 5.

43. $5 + 4x$

44. $x^2 - y$

45. $3x + 2y - 4z$

46. $\dfrac{7y^2 + 2z - 1}{x}$

In 47-50, write an algebraic expression for the each of the following. Use x, to represent "a number". Evaluate the expression when x = 2.

47. The sum of a number and (–5).

48. 4 less than a number.

49. The quotient of (–8) and a number.

50. Twice a number increased by 3.

CHAPTER 3

FRACTIONS

3.1 An Introduction to Fractions

Fractions tend to be one topic that many students dread. If this is you, we hope to change your opinion about fractions. If we take a look at what a fraction is, there is really nothing to worry about. A fraction is part of an integer. For example, if you order a large pizza to share with your roommates, and the pizza has twelve slices, each slice is considered to be $\frac{1}{12}$ of the pizza. If you eat five slices, you would be eating $\frac{5}{12}$ of the pizza. What would happen if you opened your pizza box and one of the slices was missing? This could be represented as $-\frac{1}{12}$, which means we are minus one of twelve slices.

The first thing we need to know about fractions, is the different parts of a fraction. There are three parts of a fraction: numerator, denominator, and fraction bar. The **numerator** is the number or expression on the top. The **denominator** is the number or expression on the bottom. (To help you remember this, denominator starts with "d" for "down.") The line in the fraction is known as the **fraction bar**, which means to divide.

$$\frac{numerator}{denominator} \leftarrow fraction\ bar$$

A fraction whose numerator is less than the denominator is called a **proper fraction**. The following are examples of proper fractions.

$$\frac{1}{2}, \quad -\frac{3}{4}, \quad \frac{7}{8}, \quad \text{and} \quad -\frac{5}{6}$$

A fraction whose numerator is greater than or equal to the denominator is called an **improper fraction**. The following are examples of improper fractions.

$$-\frac{11}{7}, \quad \frac{10}{10}, \quad -\frac{4}{2}, \quad \text{and} \quad \frac{3}{1}$$

One of the more common improper fractions we will see is an integer. An integer can be written as an improper fraction by placing a 1 in the denominator of the number. For example, the number 5 can be written:

$$5 = \frac{5}{1}$$

An important concept to remember when working with fractions is the fact that the **denominator** of a fraction cannot equal **0**. Division by 0 is undefined as explained in Chapter 1.

$$\frac{5}{0} = undefined \qquad \text{and} \qquad \frac{-2}{0} = undefined$$

When the numerator of a fraction is a zero, the value of the fraction is 0. Division into zero always equals 0.

$$\frac{0}{3}=0 \qquad \text{and} \qquad \frac{0}{-9}=0$$

Any improper fraction can take on another form. It can be written as a **mixed number**. A mixed number is made up of an integer and a proper fraction. The following are examples of mixed numbers:

$$1\frac{1}{3}, \qquad 7\frac{6}{9}, \qquad -3\frac{2}{5}, \qquad \text{and} \qquad 4\frac{1}{8}$$

In a mixed number, the integer and the fraction are added together. For example:

$$1\frac{1}{3}=1+\frac{1}{3} \qquad \text{and} \qquad -2\frac{1}{4}=-\left(2+\frac{1}{4}\right)$$

Regardless of how a fraction may be written, if it is **negative**, it remains **negative**. All negative fractions can be rewritten and (–1) multiplied by the fraction. A negative fraction can take on three different forms:

$$-\frac{1}{2} = \frac{-1}{2} = \frac{1}{-2}$$

The negative may be found in front of the fraction, in the numerator, or in the denominator. All three fractions shown above are equivalent.

Example 1

Identify the following as a proper fraction, improper fraction, or a mixed number.

a) $2\frac{1}{2}$ $\boxed{\textit{mixed number}}$ *There is a **whole number** and a **fraction**.*

b) $-\frac{5}{8}$ $\boxed{\textit{proper fraction}}$ *The numerator is **smaller** than the denominator.*

c) $\frac{12}{7}$ $\boxed{\textit{improper fraction}}$ *The numerator is **larger** than the denominator.*

d) $\frac{3}{5}$ $\boxed{\textit{proper fraction}}$ *The numerator is **smaller** than the denominator.*

e) $\frac{1}{5}$ $\boxed{\textit{proper fraction}}$ *The numerator is **smaller** than the denominator.*

f) $9\frac{6}{11}$ $\boxed{\textit{mixed number}}$ *There is a **whole number** and a **fraction**.*

g) $-\frac{13}{13}$ $\boxed{\textit{improper fraction}}$ *The numerator is **equal** to the denominator.*

h) $-3\frac{4}{5}$ $\boxed{\textit{mixed number}}$ *There is a **whole number** and a **fraction.***

Now take a closer look at improper fractions and mixed numbers. An improper fraction can be rewritten as a mixed number and a mixed number can be rewritten as an improper fraction.

CHANGING AN IMPROPER FRACTION INTO A MIXED NUMBER

To change an improper fraction into a mixed number we can write the fraction long-handed. This is done by rewriting the numerator of the fraction as a sum of 1's, over the denominator. Try this with the following fraction:

$$\frac{13}{5}=\frac{1+1+1+1+1+1+1+1+1+1+1+1+1}{5}$$

After rewriting the fraction, look at the denominator. The denominator tells you how many 1's it takes to make a whole number. In our example we need five 1's for each integer. Circle each group of five.

$$\frac{13}{5}=\frac{\boxed{1+1+1+1+1}+\boxed{1+1+1+1+1}+1+1+1}{5}$$

There are two groups circled. Therefore, the integer part of our mixed number is 2. There are three 1's remaining, so the fraction part of the mixed number has a numerator of 3 and the denominator stay the same, 5. Giving us the mixed number:

$$\boxed{2\frac{3}{5}}$$

Example 2 **Change the following improper fractions into mixed numbers doing it long-hand.**

a)
$$\frac{11}{4}=\frac{1+1+1+1+1+1+1+1+1+1+1}{4}$$
$$=\frac{\boxed{1+1+1+1}+\boxed{1+1+1+1}+1+1+1}{4}$$
$$=\boxed{2\frac{3}{4}}$$

Our denominator is 4, look for groups of 4.

b)
$$-\frac{7}{5}=-\frac{1+1+1+1+1+1+1}{5}$$
$$=-\frac{\boxed{1+1+1+1+1}+1+1}{5}$$
$$=\boxed{-1\frac{2}{5}}$$

Our denominator is 5, look for groups of 5.

The fraction is negative and will remain negative.

Now that we can change an improper fraction into a mixed number, examine the following fraction:

$$\frac{12}{4}=\frac{1+1+1+1+1+1+1+1+1+1+1+1}{4}$$
$$=\frac{\boxed{1+1+1+1}+\boxed{1+1+1+1}+\boxed{1+1+1+1}}{4}$$

Notice, after writing it out long-hand, there are no 1's remaining. When this happens, our improper fraction becomes an integer. In the example above, we can use the fraction bar, which means divide, to divide 12 by 4 giving us a quotient of 3. This brings us to another method we can use to change an improper fraction to a mixed number, division.

To use division, divide the numerator by the denominator. The quotient you get is the integer part of your mixed number. The remainder is the numerator of the fraction part, over the original denominator. Look at the following example:

$\frac{13}{3}$ Use long division to divide 13 by 3:

$$\begin{array}{r} 4 \leftarrow \textit{quotient} \\ 3\overline{)13} \\ -12 \\ \hline 1 \leftarrow \textit{remainder} \end{array}$$

the mixed number is: $\boxed{4\frac{1}{3}}$

Example 3

Change the following improper fractions into mixed numbers using division.

a) $\dfrac{5}{3}$

$$3\overline{)\,5}\quad \text{quotient } 1,\quad 5-3=2 \leftarrow \textit{remainder}$$

$$= \boxed{1\tfrac{2}{3}}$$

b) $\dfrac{64}{4}$

$$4\overline{)\,64}\quad \text{quotient } 16,\quad 64-4=24,\ 24-24=0 \leftarrow \textit{remainder}$$

$$= \boxed{16}$$

c) $-\dfrac{13}{9}$

$$9\overline{)\,13}\quad \text{quotient } 1,\quad 13-9=4 \leftarrow \textit{remainder}$$

$$= \boxed{-1\tfrac{4}{9}}$$

The fraction is negative and will remain negative.

CHANGING A MIXED NUMBER INTO AN IMPROPER FRACTION

When we changed an improper fraction into a mixed number, we used division. To change a mixed number into an improper fraction, we will use the opposite operation, multiplication.

To change an improper fraction into a mixed number, multiply the denominator by the integer. Take that product and add the numerator. This result is the numerator of the improper fraction. The denominator stays the same. The following illustrates this concept:

$$9\frac{1}{2} = 9\frac{1}{2} = \frac{2\cdot 9+1}{2} = \frac{18+1}{2} = \boxed{\frac{19}{2}}$$

Example 4

Change the following mixed numbers into improper fractions.

a) $$4\frac{7}{10} = 4\frac{7}{10} = \frac{10\cdot 4+7}{10} = \frac{40+7}{10} = \boxed{\frac{47}{10}}$$

b) $$-5\frac{1}{6} = -\left(5\frac{1}{6}\right) = -\left(\frac{6\cdot 5+1}{6}\right) = -\left(\frac{30+1}{6}\right) = \boxed{-\frac{31}{6}}$$

The fraction is negative and will remain negative.
Keep the negative sign in the front.

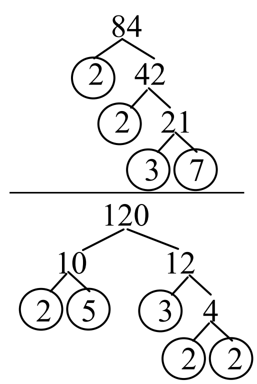
Now that we have learned how to change an improper fraction into a mixed number and a mixed number into an improper fraction, we must determine when it is appropriate to use each form of a fraction. In mathematics, it is most useful to leave your answer as an improper fraction. Improper fractions are necessary to perform mathematical computations. But, for everyday use, people tend to understand mixed numbers better. As a general rule, leave your answer as an improper fraction unless we are dealing with an application problem where it would make more sense to write your answer as a mixed number.

REDUCING FRACTIONS

Have you ever heard of "**reducing**" or "**simplifying fractions**"? What exactly does this mean? Reducing a fraction means to find an equivalent fraction in which the numerator and denominator are as small as possible. This means that there is no number, except 1, that can be divided evenly into both the numerator and the denominator. Therefore, reducing a fraction is the process of eliminating any factors the numerator and denominator have in common. This process is also called "**writing a fraction in lowest terms**".

Reducing a fraction can be done with the use of a factor tree. Use a factor tree to rewrite the numerator as a product of prime numbers. Do the same thing for the denominator. For example:

$$\frac{84}{120} =$$



$$= \frac{2\cdot2\cdot3\cdot7}{2\cdot2\cdot2\cdot3\cdot5}$$

For the next step, we need to remember, when you multiply by a number one, it does not change the value of your number. **Any expression divided by itself is equal to 1.** For example:

$$\frac{2}{2}=1 \qquad \frac{5}{5}=1 \qquad \frac{10}{10}=1 \qquad \frac{x}{x}=1,$$

and so on.

Now let's look for the ones in our fraction, $\frac{84}{120}$.

$$\frac{84}{120}=\frac{2\cdot2\cdot3\cdot7}{2\cdot2\cdot2\cdot3\cdot5}=\frac{2}{2}\cdot\frac{2}{2}\cdot\frac{3}{3}\cdot\frac{7}{2\cdot5}$$

$$= 1\cdot1\cdot1\cdot\frac{7}{10} = \boxed{\frac{7}{10}}$$

We can shorten the process of reducing by using what we refer to as "**the long bar"** method. This method is done by writing a long bar. We then place the prime factors of the numerator on the top and the prime factors of the denominator on the bottom. Make sure that everything is a product. We then find the ones.

$$\frac{60}{54}=\frac{2\cdot 2\cdot 3\cdot 5}{2\cdot 3\cdot 3\cdot 3}$$

$$=\frac{\overset{1}{\cancel{2}}\cdot 2\cdot \overset{1}{\cancel{3}}\cdot 5}{\underset{1}{\cancel{2}}\cdot \underset{1}{\cancel{3}}\cdot 3\cdot 3}$$

$$=\frac{2\cdot 5}{3\cdot 3}=\boxed{\frac{10}{9}}$$

What happens once we have found all the ones in the long bar method and there are no numbers remaining in the numerator or denominator? Look at the following examples:

$$\frac{15}{30}=\frac{3\cdot 5}{2\cdot 3\cdot 5}=\frac{?}{2} \qquad \textit{and} \qquad \frac{350}{35}=\frac{2\cdot 5\cdot 5\cdot 7}{5\cdot 7}=\frac{2\cdot 5}{?}=\frac{10}{?}$$

You may think that the fraction goes to zero because there is nothing left. (Recall, that you can't divide by zero.) This is not the case. When we have found our ones, as in $\frac{5}{5}$, the number does not disappear. The $\frac{5}{5}$ is actually replaced with its value, **1**. Therefore, if there are no remaining numbers, we are left with a 1 in that place. So,

$$\frac{15}{30}=\frac{\overset{1}{\cancel{3}}\cdot \overset{1}{\cancel{5}}}{2\cdot \underset{1}{\cancel{3}}\cdot \underset{1}{\cancel{5}}}=\boxed{\frac{1}{2}} \qquad \textit{and} \qquad \frac{350}{35}=\frac{2\cdot \overset{1}{\cancel{5}}\cdot 5\cdot \overset{1}{\cancel{7}}}{\underset{1}{\cancel{5}}\cdot \underset{1}{\cancel{7}}}=\frac{2\cdot 5}{1}=\frac{10}{1}=\boxed{10}$$

↑ *This should be written as an integer, 10.*

Example 5 **Reduce or simplify the following fractions to lowest terms.**

$$\frac{12}{16}=\frac{2\cdot 2\cdot 3}{2\cdot 2\cdot 2\cdot 2}=\frac{\overset{1}{\cancel{2}}\cdot \overset{1}{\cancel{2}}\cdot 3}{\underset{1}{\cancel{2}}\cdot \underset{1}{\cancel{2}}\cdot 2\cdot 2}$$

$$=\frac{3}{2\cdot 2}$$

$$=\boxed{\frac{3}{4}}$$

Example 6 **Reduce or simplify the following fractions to lowest terms.**

$$\frac{8}{15} = \frac{2\cdot 2\cdot 2}{3\cdot 5}$$

There are no 1's. The fraction is already in lowest terms.

$$= \boxed{\frac{8}{15}}$$

Example 7 **Reduce or simplify the following fractions to lowest terms.**

$$\frac{25}{75} = \frac{5\cdot 5}{3\cdot 5\cdot 5} = \frac{\overset{1}{\cancel{5}}\cdot\overset{1}{\cancel{5}}}{3\cdot\underset{1}{\cancel{5}}\cdot\underset{1}{\cancel{5}}}$$

$$= \boxed{\frac{1}{3}}$$

Example 8 **Reduce or simplify the following fractions to lowest terms.**

$$-\frac{96}{124} = \frac{2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 3}{2\cdot 2\cdot 31} = -\frac{\overset{1}{\cancel{2}}\cdot\overset{1}{\cancel{2}}\cdot 2\cdot 2\cdot 2\cdot 3}{\underset{1}{\cancel{2}}\cdot\underset{1}{\cancel{2}}\cdot 31}$$

$$= -\frac{2\cdot 2\cdot 2\cdot 3}{31}$$

$$= \boxed{-\frac{24}{31}}$$ *The fraction is negative and will remain negative.*

The long bar method of reducing works with numbers as well as with variables. When a variable occurs in a fraction, write it in expanded form. Look for the ones in our fraction and simplify in the same manner we do with numbers. In this chapter, we will only use the long bar method when variables occur in the problem. In Chapter 9, a more in-depth discussion of reducing fractions with variables will be given.

Note: Fractions are undefined if the denominator is equal to zero. For this reason, we will assume that all variables in this chapter are not equal to zero.

Example 9

Reduce or simplify the following fractions to lowest terms.

$$\frac{15x^3}{25x} = \frac{3\cdot 5\cdot x\cdot x\cdot x}{5\cdot 5\cdot x} = \frac{3\cdot \overset{1}{\cancel{5}}\cdot \overset{1}{\cancel{x}}\cdot x\cdot x}{\underset{1}{\cancel{5}}\cdot 5\cdot \underset{1}{\cancel{x}}}$$

$$= \frac{3\cdot x\cdot x}{5}$$

$$= \boxed{\frac{3x^2}{5}}$$

Example 10

Reduce or simplify the following fractions to lowest terms.

$$\frac{36x^2y^4}{9x^5y^2} = \frac{2\cdot 2\cdot 3\cdot 3\cdot x\cdot x\cdot y\cdot y\cdot y\cdot y}{3\cdot 3\cdot x\cdot x\cdot x\cdot x\cdot x\cdot y\cdot y}$$

$$= \frac{2\cdot 2\cdot \overset{1}{\cancel{3}}\cdot \overset{1}{\cancel{3}}\cdot \overset{1}{\cancel{x}}\cdot \overset{1}{\cancel{x}}\cdot \overset{1}{\cancel{y}}\cdot \overset{1}{\cancel{y}}\cdot y\cdot y}{\underset{1}{\cancel{3}}\cdot \underset{1}{\cancel{3}}\cdot \underset{1}{\cancel{x}}\cdot \underset{1}{\cancel{x}}\cdot x\cdot x\cdot x\cdot \underset{1}{\cancel{y}}\cdot \underset{1}{\cancel{y}}}$$

$$= \frac{2\cdot 2\cdot y\cdot y}{x\cdot x\cdot x}$$

$$= \boxed{\frac{4y^2}{x^3}}$$

In the introduction to reducing fractions it was mentioned that a fraction in lowest terms meant that there is no number, except 1, that can be divided evenly into both the numerator and the denominator. Using this concept, a fraction can be reduced using division. When the numerator and denominator of a fraction share a common factor, we can reduce by dividing both parts of the fraction by their greatest common factor. The following two examples will demonstrate how to reduce a fraction using division.

Example 11

Reduce or simplify the following fractions to lowest terms using division.

$$\frac{10}{25}$$ *10 and 25 share a common factor of 5.*

$$\frac{10^{\div 5}}{25_{\div 5}}$$ *Divide the numerator and denominator by 5.*

$$\boxed{\frac{2}{5}}$$

Example 12

Reduce or simplify the following fractions to lowest terms using division.

$\frac{56}{64}$

56 and 64 share common factors of 2, 4, and 8.

$\frac{56^{\div 8}}{64_{\div 8}}$

To reduce to lowest terms, divide the numerator and denominator by the greatest common factor, 8.

$\boxed{\frac{7}{8}}$

We can reduce a fraction by dividing both the numerator and denominator by any common factor. This will not necessarily reduce the fraction to lowest terms. If we reduce by dividing a common factor that is not the greatest common factor, we will need to reduce again with another common factor. We can repeat this process until the only common factor that remains is 1. Examine Example 13 for this technique.

Example 13

Reduce or simplify the following fractions to lowest terms using division.

$\frac{96}{120}$

96 and 120 share common factors of 2. Although 2 is not the greatest common factor, we can reduce the fraction by dividing by 2.

$\frac{96^{\div 2}}{120_{\div 2}}$

$\frac{48^{\div 2}}{60_{\div 2}}$

We now have a fraction equivalent to the original fraction. This new fraction is not completely reduced 48 and 60 share common factors. 2 is a common factor of 48 and 60. Reduce the fraction by dividing by 2.

$\frac{24^{\div 2}}{30_{\div 2}}$

We again have an equivalent fraction but it is not completely reduced. 24 and 30 share a common factor of 2. Reduce the fraction by dividing by 2.

$\frac{12^{\div 3}}{15_{\div 3}}$

Again we have another equivalent fraction. Yet, it is still not completely reduced. 12 and 15 share a common factor of 3. Reduce the fraction by dividing by 3.

$\boxed{\frac{4}{5}}$

The equivalent fraction is now in lowest terms because the only common factor of 4 and 5 remaining is 1.

Example 13 could have been reduced quicker by dividing by the greatest common factor of 24 initially. Always check the final answer to make sure the numerator and denominator don't share a common factor other than 1.

3.1 EXERCISES

In 1-6, identify the following as a proper fraction, improper fraction, or mixed number.

1. $-\frac{3}{5}$
2. $113\frac{1}{3}$
3. $-\frac{12}{5}$
4. $\frac{17}{17}$
5. $\frac{4}{9}$
6. $9\frac{7}{8}$

In 7-12, change the following improper fractions into mixed numbers using division.

7. $\frac{8}{3}$
8. $-\frac{13}{4}$
9. $\frac{25}{5}$
10. $\frac{35}{6}$
11. $-\frac{100}{9}$
12. $\frac{111}{4}$

In 13-18, change the following mixed numbers into improper fractions.

13. $-7\frac{2}{3}$
14. $3\frac{5}{6}$
15. $2\frac{11}{12}$
16. $-8\frac{3}{5}$
17. $15\frac{1}{4}$
18. $-3\frac{11}{25}$

In 19-28, reduce or simplify the following fractions to lowest terms.

19. $\dfrac{16}{20}$

20. $-\dfrac{60}{42}$

21. $\dfrac{x^5}{x^7}$

22. $\dfrac{11}{13}$

23. $\dfrac{125}{70}$

24. $\dfrac{100}{28}$

25. $\dfrac{30}{81}$

26. $-\dfrac{21a^6}{63a^3}$

27. $-\dfrac{15x^9y^4}{125x^6y^7}$

28. $\dfrac{4a^3b^2c^4d^0}{16ab^2c^3}$

In 29-30, solve the following application problems.

29. In a math class there are sixteen females. The total number of students in the class is forty. Write a fraction of the total number of students that are female. Reduce if possible.

30. Tony's girlfriend has made him a banana cream pie. Tony cut the pie into eight pieces and ate three of the pieces. Represent the number of pieces that were eaten of the total pieces as a fraction. Reduce if possible.

3.2 Multiplication of Fractions

Your English professor asks you how many hours per week you spend outside of class reading. You spend $\frac{1}{2}$ hour per day reading. How many hours per week will you tell your professor you spend reading?

There are seven days in one week so we take $\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}$, which gives us seven halves and can be written as $\frac{7}{2}$. In this example, there is a shorter way to simplify the problem. It can be done with the use of multiplication. There are 7 days and you read $\frac{1}{2}$ hour each day. Multiply 7 by $\frac{1}{2}$. To do this, write 7 as an improper fraction, $\frac{7}{1}$, then multiply. Multiply the numerators together, then, multiply the denominators together.

$$\frac{7}{1}\cdot\frac{1}{2} = \frac{7\cdot 1}{1\cdot 2} = \frac{7}{2}$$

MULTIPLICATION OF FRACTIONS

To multiply fractions, multiply the numerators together, then, multiply the denominators together. (*b* and *d* are not equal to zero.)

$$\frac{a}{b}\cdot\frac{c}{d} = \frac{a\cdot c}{b\cdot d}$$

When multiplying fractions, we need to make sure our answer is in lowest terms. After we find the product, we will reduce the fraction using "the long bar method." (Remember to use a factor tree to find the prime numbers.)

Example 1

Find the product. Reduce to lowest terms. $\frac{8}{15}\cdot\frac{25}{28}$

$$\frac{8}{15}\cdot\frac{25}{28} = \frac{8\cdot 25}{15\cdot 28} = \frac{200}{420}$$

$$\frac{200}{420} = \frac{\overset{1}{\cancel{2}}\cdot\overset{1}{\cancel{2}}\cdot 2\cdot\overset{1}{\cancel{5}}\cdot 5}{\underset{1}{\cancel{2}}\cdot\underset{1}{\cancel{2}}\cdot 3\cdot\underset{1}{\cancel{5}}\cdot 7} = \frac{2\cdot 5}{3\cdot 7}$$

$$= \boxed{\frac{10}{21}}$$

Example 2 **Find the product. Reduce to lowest terms.** $\frac{20}{21}\cdot\frac{7}{10}$

$$\frac{20}{21}\cdot\frac{7}{10} = \frac{20\cdot 7}{21\cdot 10} = \frac{140}{210}$$

$$\frac{140}{210} = \frac{\overset{1}{\cancel{2}}\cdot 2\cdot \overset{1}{\cancel{5}}\cdot \overset{1}{\cancel{7}}}{\underset{1}{\cancel{2}}\cdot 3\cdot \underset{1}{\cancel{5}}\cdot \underset{1}{\cancel{7}}}$$

$$= \boxed{\frac{2}{3}}$$

To shorten this process, we can use the "long bar" before we find the product.

$$\frac{20}{21}\cdot\frac{7}{10} = \frac{2\cdot 2\cdot 5\ \cdot\ 7}{3\cdot 7\ \cdot\ 2\cdot 5}$$

$$= \frac{\overset{1}{\cancel{2}}\cdot 2\cdot \overset{1}{\cancel{5}}\cdot \overset{1}{\cancel{7}}}{3\cdot \underset{1}{\cancel{7}}\cdot \underset{1}{\cancel{2}}\cdot \underset{1}{\cancel{5}}}$$

$$= \boxed{\frac{2}{3}}$$

In multiplication of fractions, there is another way to reduce the fraction. This is done before we find the product. This is commonly known as "**cross-canceling**" and can **only** be done with multiplication of fractions. To cross-cancel, look for a common factor in the numerator and denominator of either fraction. Then divide each number by that common factor. (You can't divide by a common factor if both numbers are in the numerator or both numbers are in the denominator.) Find the product of the remaining numbers. This concept is illustrated in the example below.

$$\frac{20^{\div 10}}{21}\cdot\frac{7}{10_{\div 10}} = \frac{2}{21}\cdot\frac{7}{1}$$ *The 20 & 10 share a factor of 10. Divide each by 10.*

$$= \frac{2}{21_{\div 7}}\cdot\frac{7^{\div 7}}{1}$$ *The 21 & 7 share a factor of 7. Divide each by 7.*

$$= \frac{2}{3}\cdot\frac{1}{1}$$ *Now, find the product.*

$$= \boxed{\frac{2}{3}}$$

MULTIPLYING MIXED NUMBERS

To multiply mixed numbers, we must first change the mixed numbers into improper fractions. Then multiply using any of the methods we have discussed previously.

$$2\frac{2}{3}\cdot 5\frac{3}{4} = \frac{8}{3}\cdot\frac{23}{4} =$$

"long bar" $\quad \frac{8}{3}\cdot\frac{23}{4} = \frac{\overset{1}{\cancel{2}}\cdot\overset{1}{\cancel{2}}\cdot 2\cdot 23}{3\cdot\underset{1}{\cancel{2}}\cdot\underset{1}{\cancel{2}}} = \boxed{\frac{46}{3}}$

"cross canceling" $\quad \frac{\overset{2}{\cancel{8}}}{3}\cdot\frac{23}{\underset{1}{\cancel{4}}} = \boxed{\frac{46}{3}}$

After looking at the different ways to multiply and reduce a fraction, it is important that you choose which method works best for you. It is also important to remember the rules for multiplication of signed numbers.

like signs	positive $\times/\div$ positive = positive
	negative $\times/\div$ negative = positive
unlike signs	positive $\times/\div$ negative = negative
	negative $\times/\div$ posititve = negative

All of the previous examples have demonstrated finding the product of two fractions. The methods of multiplication do not depend on how many fractions we find the product of. We can use these methods to find the product of several fractions at one time.

Example 3 **Find the product. Reduce to lowest terms.** $\frac{3}{4}\cdot\frac{2}{9}\cdot\frac{3}{5}$

$$\frac{3}{4}\cdot\frac{2}{9}\cdot\frac{3}{5} =$$

"long bar" $\quad \frac{3}{4}\cdot\frac{2}{9}\cdot\frac{3}{5} = \frac{\overset{1}{\cancel{3}}\cdot\overset{1}{\cancel{2}}\cdot\overset{1}{\cancel{3}}}{\underset{1}{\cancel{2}}\cdot 2\cdot\underset{1}{\cancel{3}}\cdot\underset{1}{\cancel{3}}\cdot 5} = \boxed{\frac{1}{10}}$

"cross canceling" $\quad \frac{\overset{1}{\cancel{3}}}{\underset{2}{\cancel{4}}}\cdot\frac{\overset{1}{\cancel{2}}}{\underset{\underset{1}{\cancel{3}}}{\cancel{9}}}\cdot\frac{\overset{1}{\cancel{3}}}{5} = \boxed{\frac{1}{10}}$

When working with variables, we suggest using the "long bar" method. In Chapter 9, we will discuss another method of simplifying with fractions. Recall: Fractions are undefined if the denominator is equal to zero. For this reason, we will assume that all variables in this chapter are not equal to zero.

Example 4

Find the product. Reduce to lowest terms. $\frac{x^2y}{z}\cdot\frac{yz^3}{x^5}$

$$\frac{x^2y}{z}\cdot\frac{yz^3}{x^5} = \frac{\overset{1}{\cancel{x}}\cdot\overset{1}{\cancel{x}}\cdot y\cdot y\cdot\overset{1}{\cancel{z}}\cdot z\cdot z}{\underset{1}{\cancel{z}}\cdot\underset{1}{\cancel{x}}\cdot\underset{1}{\cancel{x}}\cdot x\cdot x\cdot x}$$

$$= \frac{y\cdot y\cdot z\cdot z}{x\cdot x\cdot x}$$

$$= \boxed{\frac{y^2z^2}{x^3}}$$

Example 5

Find the product. Reduce to lowest terms. $\left(-\frac{1}{5}\right)^3$

$$\left(-\frac{1}{5}\right)^3 = \left(-\frac{1}{5}\right)\cdot\left(-\frac{1}{5}\right)\cdot\left(-\frac{1}{5}\right)$$

$$= \frac{-1\cdot -1\cdot -1}{5\cdot 5\cdot 5}$$

$$= \boxed{-\frac{1}{125}}$$

The product of three negatives is a negative. Your final answer will be negative.

Example 6

Find the product. Reduce to lowest terms. $7\frac{1}{7}\cdot 8\cdot 6\frac{3}{10}$

$$7\frac{1}{7}\cdot 8\cdot 6\frac{3}{10} = \frac{50}{7}\cdot\frac{8}{1}\cdot\frac{63}{10}$$

Start by rewriting the mixed numbers as improper fractions.

$\frac{50}{7}\cdot\frac{8}{1}\cdot\frac{63}{10} =$

"long bar"

$$\frac{50}{7}\cdot\frac{8}{1}\cdot\frac{63}{10} = \frac{\overset{1}{\cancel{2}}\cdot\overset{1}{\cancel{5}}\cdot 5\cdot 2\cdot 2\cdot 2\cdot 3\cdot 3\cdot\overset{1}{\cancel{7}}}{\underset{1}{\cancel{7}}\cdot 1\cdot\underset{1}{\cancel{2}}\cdot\underset{1}{\cancel{5}}} = \frac{360}{1} = \boxed{360}$$

"cross canceling"

$$\frac{\overset{5}{\cancel{50}}}{\underset{1}{\cancel{7}}}\cdot\frac{8}{1}\cdot\frac{\overset{9}{\cancel{63}}}{\underset{1}{\cancel{10}}} = \frac{360}{1} = \boxed{360}$$

Example 7

Find the product. Reduce to lowest terms. $-\dfrac{15x^2y}{32x}\cdot-\dfrac{8xy}{5yz}$

$$-\frac{15x^2y}{32x}\cdot-\frac{8xy}{5yz} = \frac{3\cdot\cancel{5}\cdot\cancel{x}\cdot x\cdot y\ \cdot\cancel{2}\cdot\cancel{2}\cdot\cancel{2}\cdot x\cdot\cancel{y}}{\cancel{2}\cdot\cancel{2}\cdot\cancel{2}\cdot 2\cdot 2\ \cdot\cancel{5}\cdot\cancel{y}\cdot z}$$

A negative multiplied by a negative, results in a positive answer.

$$= \frac{3\cdot x\cdot x\cdot y}{2\cdot 2\cdot z}$$

$$= \boxed{\frac{3x^2y}{4z}}$$

Example 8

Find the product. Reduce to lowest terms. $\left(\dfrac{3}{4}\right)^2\cdot\left(-\dfrac{2}{3}\right)^3$

$$\left(\frac{3}{4}\right)^2\cdot\left(-\frac{2}{3}\right)^3 = \left(\frac{3}{4}\right)\left(\frac{3}{4}\right)\cdot\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)$$

$$= \left(\frac{\cancel{3}}{\cancel{2}\cdot\cancel{2}}\right)\left(\frac{\cancel{3}}{\cancel{2}\cdot 2}\right)\cdot\left(-\frac{\cancel{2}}{\cancel{3}}\right)\left(-\frac{\cancel{2}}{\cancel{3}}\right)\left(-\frac{\cancel{2}}{3}\right)$$

The product of three negatives is a negative. Your final answer will be negative.

$$= -\frac{1}{2\cdot 3}$$

$$= \boxed{-\frac{1}{6}}$$

Example 9

Multiply $\dfrac{3}{4}$ and $\dfrac{8}{11}$. Reduce to lowest terms.

$\dfrac{3}{4}\cdot\dfrac{8}{11} =$

"long bar" $\quad \dfrac{3}{4}\cdot\dfrac{8}{11} = \dfrac{3\cdot\cancel{2}\cdot\cancel{2}\cdot 2}{\cancel{2}\cdot\cancel{2}\cdot 11} = \boxed{\dfrac{6}{11}}$

"cross canceling" $\quad \dfrac{3}{\cancel{4}_1}\cdot\dfrac{\cancel{8}^2}{11} = \boxed{\dfrac{6}{11}}$

Example 10 **Find the product of** $2\frac{7}{8}, 1\frac{11}{13},$ **and** $4\frac{1}{2}$**. Reduce to lowest terms.**

$$2\frac{7}{8}\cdot 1\frac{11}{13}\cdot 4\frac{1}{2} = \frac{23}{8}\cdot\frac{24}{13}\cdot\frac{9}{2}$$

Start by rewriting the mixed numbers as improper fractions.

$$\frac{23}{8}\cdot\frac{24}{13}\cdot\frac{9}{2} =$$

"long bar" $$\frac{23}{8}\cdot\frac{24}{13}\cdot\frac{9}{2} = \frac{23\cdot \cancel{2}^{1}\cdot \cancel{2}^{1}\cdot \cancel{2}^{1}\cdot 3\cdot 3\cdot 3}{\cancel{2}_{1}\cdot \cancel{2}_{1}\cdot \cancel{2}_{1}\cdot 13\cdot 2} = \boxed{\frac{621}{26}}$$

"cross canceling" $$\frac{23}{\cancel{8}_{1}}\cdot\frac{\cancel{24}^{3}}{13}\cdot\frac{9}{2} = \boxed{\frac{621}{26}}$$

Example 11 ***Application*****. In a college history class, two-elevenths of the students did <u>not</u> pass the class. If there are thirty-three students in the class, how many students did <u>not</u> pass the class? How many students passed the class?**

The word "of" means to multiply, in order to solve this problem, we must multiply two-elevenths by thirty-three.

$$\frac{2}{11}\cdot 33 = \frac{2}{11}\cdot\frac{33}{1}$$

Start by rewriting the whole number as an improper fraction.

$$\frac{2}{11}\cdot\frac{33}{1} =$$

"long bar" $$\frac{2}{11}\cdot\frac{33}{1} = \frac{2\cdot 3\cdot \cancel{11}^{1}}{\cancel{11}_{1}\cdot 1} = \frac{6}{1} = \boxed{6}$$

"cross canceling" $$\frac{2}{\cancel{11}_{1}}\cdot\frac{\cancel{33}^{3}}{1} = \frac{6}{1} = \boxed{6}$$

The number of students that did <u>not</u> pass the class is **6**.

The number of students that passed the class is 33 – 6 = **27**.

3.2 EXERCISES

In 1-18, multiply the following fractions. Reduce all answers to lowest terms.

1. $\frac{2}{3}\cdot\frac{7}{5}$

2. $4\cdot\frac{5}{2}$

3. $-\frac{6}{7}\cdot\frac{2}{5}$

4. $\frac{7}{8}\cdot\frac{15}{16}$

5. $\frac{14}{45}\cdot\frac{-3}{56}$

6. $10\cdot\frac{1}{100}\cdot\frac{1}{10}$

7. $\frac{x}{y}\cdot\frac{y^2}{x}$

8. $\frac{3}{5}\cdot\frac{2}{9}$

9. $\frac{ab}{c}\cdot\frac{c^2}{b^2}$

10. $\frac{64x^2}{9y^2}\cdot\frac{72y}{16x}$

11. $\frac{-55}{19}\cdot\frac{-38}{11}$

12. $11\frac{5}{6}\cdot 3\frac{1}{2}$

13. $\frac{5}{16}\cdot 84$

14. $-1\frac{2}{3}\cdot 3\frac{3}{5}$

15. $\frac{1}{2}\cdot\frac{5}{6}\cdot\frac{8}{10}$

16. $\left(\frac{4}{3}\right)^3$

17. $\left(1\frac{1}{4}\right)^2$

18. $\frac{7}{8}\cdot\frac{2}{5}\cdot 6\cdot\frac{15}{7}$

In 19-22, perform the indicated operation. Reduce all answers to lowest terms.

19. Find the product of $\frac{10}{13}$ and $\frac{3}{5}$.

20. Find the product of $10\frac{2}{3}$ and $8\frac{1}{2}$.

21. Find $\frac{1}{2}$ of $\frac{10}{3}$.

22. Find $\frac{3}{5}$ of $\frac{1}{3}$ of 15.

In 23-26, solve the following application problems.

23. A cook needs to triple a recipe that uses three and one-half cups of sugar. How much sugar will be needed?

24. A truck can carry seven-eighths ton of gravel. How many tons of gravel are in five loads?

25. In a history class five-sixths of the students passed. If there are thirty-six students in the class, how many students passed the class? How many students did not pass the class?

26. Hanna addresses three-eighths of a box of wedding invitations each day. At this rate, how many boxes will she complete in four and two-thirds days?

In 27-28, read and respond to the following exercises.

27. Do you prefer cross-canceling or the long bar method? Why?

28. What do you do to a mixed number before you can multiply it with another fraction?

3.3 Reciprocals & Division of Fractions

RECIPROCALS

The Multiplicative Inverse Property states that the product of any non-zero real number and its multiplicative inverse is one.

> **MULTIPLICATIVE INVERSE PROPERTY**
>
> For all non-zero real numbers *a,*
>
> $$a\cdot\left(\frac{1}{a}\right)=1 \quad \text{and} \quad \left(\frac{1}{a}\right)\cdot a=1$$

The multiplicative inverse is also known as the reciprocal of a number. For example:

$$4\cdot\left(\frac{1}{4}\right)=1 \quad \text{and} \quad \left(-\frac{3}{7}\right)\cdot\left(-\frac{7}{3}\right)=1$$

When dividing fractions, we use what is known as a reciprocal. **Reciprocals** are two numbers that have a product of 1. The reciprocal of a fraction is written by flipping or inverting the fraction. For instance, the reciprocal of $\frac{3}{8}$ is $\frac{8}{3}$, the reciprocal of $\frac{-7}{8}$ is $\frac{-8}{7}$, and the reciprocal of $\frac{1}{4}$ is $\frac{4}{1}$, which can also be written as 4. When a fraction is negative, it's reciprocal is also negative. When a fraction is positive, it's reciprocal is also positive.

To find the reciprocal of a mixed number or a whole number, we must first make it an improper fraction and then invert it. For example:

$$2\frac{1}{5} = 2\overset{+}{\underset{\cdot}{<}}\frac{1}{5} = \frac{11}{5} \text{ so the reciprocal of } 2\frac{1}{5} \text{ is } \frac{5}{11}.$$

In order for two numbers to be reciprocals, their **product must equal 1.**

$\frac{4}{5}$ and $\frac{5}{4}$ are reciprocals because $\frac{4}{5}\cdot\frac{5}{4}=\frac{20}{20}=1$

$-\frac{7}{4}$ and $-\frac{4}{7}$ are reciprocals because $-\frac{7}{4}\cdot-\frac{4}{7}=\frac{28}{28}=1$

$\frac{1}{5}$ and 5 are reciprocals because $\frac{1}{5}\cdot 5=\frac{1}{5}\cdot\frac{5}{1}=\frac{5}{5}=1$

$-8\frac{1}{6}$ and $-\frac{6}{49}$ are reciprocals because $-8\frac{1}{6}\cdot-\frac{6}{49}=-\frac{49}{6}\cdot-\frac{6}{49}=\frac{294}{294}=1$

$\frac{x}{2}$ and $\frac{2}{x}$ are reciprocals because $\frac{x}{2} \cdot \frac{2}{x} = \frac{2x}{2x} = 1$

Zero does not have a reciprocal because you cannot multiply by zero and get a product equal to one. For example:

$\frac{0}{5}$ and $\frac{5}{0}$ are **not** reciprocals because $\frac{0}{5} \cdot \frac{5}{0} \neq 1$

Recall, fractions are undefined if the denominator is equal to zero. For this reason, we will assume that all variables in this chapter are not equal to zero.

Example 1 **Write the reciprocals of the following fractions.**

	Fraction			**Reciprocal**
a)	$\frac{4}{5}$			$\boxed{\frac{5}{4}}$
b)	$\frac{3}{x^2}$			$\boxed{\frac{x^2}{3}}$
c)	-8	*Make the whole number into an improper fraction before inverting it.*	$-8 = -\frac{8}{1}$	$\boxed{-\frac{1}{8}}$
d)	$\frac{1}{7}$			$\frac{7}{1} = \boxed{7}$
e)	$-4\frac{1}{8}$	*Make the mixed number into an improper fraction before inverting it.*	$-4\frac{1}{8} = -\frac{33}{8}$	$\boxed{-\frac{8}{33}}$

DIVISION OF FRACTIONS

Nicole has $\frac{5}{2}$ tablespoons of butter. She wants to make as many cookies as possible. To make one batch of cookies, Nicole needs $\frac{3}{8}$ tablespoon of butter. How many batches of cookies can she make?

In order to solve this problem, take $\frac{5}{2}$ tablespoons and divide it by $\frac{3}{8}$ tablespoon.

Now let's learn how to divide so that we may determine the number of batches of cookies Nicole can make.

If you can multiply fractions, you should have no problem dividing fractions. Reciprocals will be used to divide fractions. Multiply the first fraction by the reciprocal of the second fraction.

> **DIVISION OF FRACTIONS**
>
> To divide fractions, multiply the first fraction by the reciprocal of the second fraction. (*b, c,* and *d* are not equal to zero.)
>
> $$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c}$$

In the example, Nicole would take:

$$\frac{5}{2} \div \frac{3}{8} = \frac{5}{2} \cdot \frac{8}{3} =$$

"long bar" $$\frac{5}{2} \cdot \frac{8}{3} = \frac{5 \cdot \overset{1}{\cancel{2}} \cdot 2 \cdot 2}{\underset{1}{\cancel{2}} \cdot 3} = \boxed{\frac{20}{3}}$$

"cross canceling" $$\frac{5}{\underset{1}{\cancel{2}}} \cdot \frac{\overset{4}{\cancel{8}}}{3} = \boxed{\frac{20}{3}}$$

Nicole could make $\frac{20}{3}$ batches of cookies. This answer would make more sense as a mixed number. Change $\frac{20}{3}$ into the mixed number $6\frac{2}{3}$. Nicole could make $6\frac{2}{3}$ batches of cookies.

Example 2 **Find the quotient. Reduce to lowest terms.** $\frac{2}{3} \div \frac{5}{6}$

$$\frac{2}{3} \div \frac{5}{6} = \frac{2}{3} \cdot \frac{6}{5} =$$

"long bar" $$\frac{2}{3} \cdot \frac{6}{5} = \frac{2 \cdot 2 \cdot \overset{1}{\cancel{3}}}{\underset{1}{\cancel{3}} \cdot 5} = \boxed{\frac{4}{5}}$$

"cross canceling" $$\frac{2}{\underset{1}{\cancel{3}}} \cdot \frac{\overset{2}{\cancel{6}}}{5} = \boxed{\frac{4}{5}}$$

Example 3 **Find the quotient. Reduce to lowest terms.** $12 \div 4\frac{7}{8}$

$$12 \div 4\frac{7}{8} = \frac{12}{1} \div \frac{39}{8} = \frac{12}{1} \cdot \frac{8}{39} =$$

"long bar" $\frac{12}{1} \cdot \frac{8}{39} = \frac{2 \cdot 2 \cdot \overset{1}{\cancel{3}} \cdot 2 \cdot 2 \cdot 2}{1 \cdot \underset{1}{\cancel{3}} \cdot 13} = \boxed{\frac{32}{13}}$

"cross canceling" $\frac{\overset{4}{\cancel{12}}}{1} \cdot \frac{8}{\underset{13}{\cancel{39}}} = \boxed{\frac{32}{13}}$

Example 4 **Find the quotient. Reduce to lowest terms.** $\frac{x}{y^2 z} \div \frac{x^2}{yz}$

When working with variables, it is best to use the "long bar" method.

$$\frac{x}{y^2 z} \div \frac{x^2}{yz} = \frac{x}{y^2 z} \cdot \frac{yz}{x^2} = \frac{\overset{1}{\cancel{x}} \cdot \overset{1}{\cancel{y}} \cdot \overset{1}{\cancel{z}}}{\underset{1}{\cancel{y}} \cdot y \cdot \underset{1}{\cancel{z}} \cdot \underset{1}{\cancel{x}} \cdot x} = \boxed{\frac{1}{xy}}$$

Example 5 **Find the quotient. Reduce to lowest terms.** $\dfrac{\frac{2}{3}}{\frac{3}{4}}$

$$\dfrac{\frac{2}{3}}{\frac{3}{4}} = \frac{2}{3} \div \frac{3}{4} = \frac{2}{3} \cdot \frac{4}{3} =$$

"long bar" $\frac{2}{3} \cdot \frac{4}{3} = \frac{2 \cdot 2 \cdot 2}{3 \cdot 3} = \boxed{\frac{8}{9}}$

"cross canceling" $\frac{2}{3} \cdot \frac{4}{3} = \boxed{\frac{8}{9}}$

Example 6

Find the quotient of $\frac{24}{25}$ and $\frac{12}{15}$. Reduce to lowest terms.

$$\frac{24}{25} \div \frac{12}{15} = \frac{24}{25} \cdot \frac{15}{12} =$$

"long bar"

$$\frac{24}{25} \cdot \frac{15}{12} = \frac{\cancel{2}^{1} \cdot \cancel{2}^{1} \cdot 2 \cdot \cancel{3}^{1} \cdot 3 \cdot \cancel{5}^{1}}{\cancel{5}_{1} \cdot 5 \cdot \cancel{2}_{1} \cdot \cancel{2}_{1} \cdot \cancel{3}_{1}} = \boxed{\frac{6}{5}}$$

"cross canceling"

$$\frac{\cancel{24}^{2}}{\cancel{25}_{5}} \cdot \frac{\cancel{15}^{3}}{\cancel{12}_{1}} = \boxed{\frac{6}{5}}$$

Example 7

Divide $\frac{18}{35}$ into $\frac{9}{14}$. Reduce to lowest terms.

The word "into" reverses the order.

$$\frac{9}{14} \div \frac{18}{35} = \frac{9}{14} \cdot \frac{35}{18} =$$

"long bar"

$$\frac{9}{14} \cdot \frac{35}{18} = \frac{\cancel{3}^{1} \cdot \cancel{3}^{1} \cdot 5 \cdot \cancel{7}^{1}}{2 \cdot \cancel{7}_{1} \cdot 2 \cdot \cancel{3}_{1} \cdot \cancel{3}_{1}} = \boxed{\frac{5}{4}}$$

"cross canceling"

$$\frac{\cancel{9}^{1}}{\cancel{14}_{2}} \cdot \frac{\cancel{35}^{5}}{\cancel{18}_{2}} = \boxed{\frac{5}{4}}$$

Example 8

***Application*. If it takes $\frac{1}{2}$ cup of sugar to make a batch of cookies, how many batches of cookies can be made from $4\frac{3}{4}$ cups of sugar?**

To solve this problem, we must divide the total amount of sugar by the amount needed to make a batch of cookies.

$$4\frac{3}{4} \div \frac{1}{2} = \frac{19}{4} \div \frac{1}{2} = \frac{19}{4} \cdot \frac{2}{1} =$$

"long bar"

$$\frac{19}{4} \cdot \frac{2}{1} = \frac{19 \cdot \cancel{2}^{1}}{\cancel{2}_{1} \cdot 2} = \boxed{\frac{19}{2}}$$

"cross canceling"

$$\frac{19}{\cancel{4}_{2}} \cdot \frac{\cancel{2}^{1}}{1} = \boxed{\frac{19}{2}}$$

$\frac{19}{2}$ *or* $9\frac{1}{2}$ batches of cookies could be made from $4\frac{3}{4}$ cups of sugar.

3.3 EXERCISES

In 1-6, find the reciprocals of the fractions.

1. $\frac{5}{8}$
2. $7\frac{2}{3}$
3. -12
4. $\frac{x}{y}$
5. $-\frac{3}{4}$
6. $-5\frac{3}{4}$

In 7-20, divide the following fractions. Reduce all answers to lowest terms.

7. $\frac{17}{20} \div 13\frac{3}{5}$
8. $\frac{11}{8} \div \frac{11}{8}$
9. $-\frac{2}{3} \div \frac{5}{7}$
10. $\frac{4}{5} \div \frac{10}{2}$
11. $\dfrac{4\frac{1}{3}}{2\frac{8}{9}}$
12. $-4 \div -\frac{2}{3}$
13. $\dfrac{7\frac{7}{8}}{9}$
14. $\frac{1}{4} \div \frac{1}{8}$
15. $\frac{ab^2}{cd} \div \frac{c^2d^2}{ab^2}$
16. $\frac{24x^4y}{35xy} \div 6x^2$

17. $1\frac{1}{3} \div 8$

18. $\dfrac{\frac{14}{3}}{\frac{16}{9}}$

19. $\dfrac{4x}{15y} \div \dfrac{21y}{100x}$

20. $\dfrac{1890x^2yz^3}{924yz} \div \dfrac{30xyz}{231x^2yz}$

In 21-24, perform the indicated operation. Reduce all answers to lowest terms.

21. Find the quotient of $\frac{5}{6}$ and $\frac{15}{30}$.

22. What is the quotient of $\frac{3}{4}$ and -3?

23. Divide $\frac{25}{36}$ by $\frac{5}{12}$.

24. Divide $28\frac{1}{2}$ by 19.

In 25-30, solve the following application problems.

25. It takes $\frac{4}{5}$ of a yard of fabric to make a pillow. How many pillows can be made from $4\frac{2}{5}$ yards of fabric?

26. How many $\frac{1}{8}$ cup measures are there in $2\frac{3}{4}$ cups?

27. Kolby earned $38\frac{1}{2}$ dollars for working 7 hours. How much did he earn per hour? Express your answer in fraction form.

28. How many bottles containing $\frac{4}{5}$ pint of liquid can be filled from a 22-pint container?

29. If a plane flies $65\frac{2}{3}$ miles in $7\frac{1}{2}$ minutes, how far can it travel in one minute?

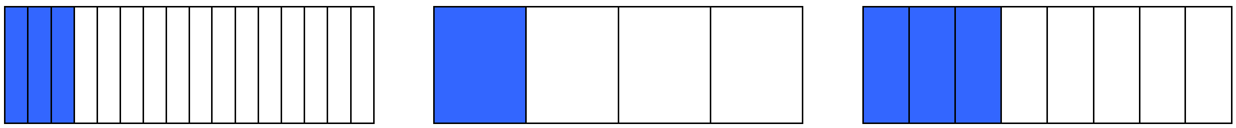
30. A cookie factory uses 1/6 of a barrel of oatmeal in each batch of cookies. The factory used 2/3 of a barrel of oatmeal yesterday. How many batches of cookies did the factory make?

3.4 Addition & Subtraction of Fractions

Now that we have learned how to multiply and divide fractions, it is time to discover how to add and subtract fractions. Examine the following example:

In a mathematics class, students must earn a C to pass the class. If $\frac{3}{16}$ earned A's, $\frac{1}{4}$ earned B's, and $\frac{3}{8}$ earned C's, how many students passed the class?

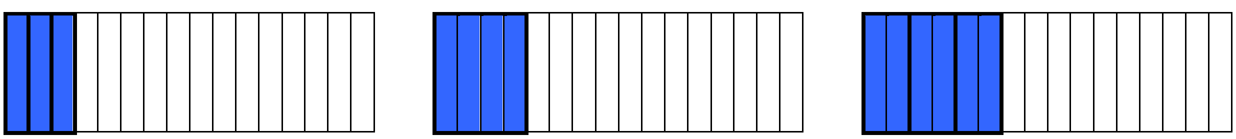
To find the number of students that passed the class, add the number of students that earned A's, B's, and C's. To do this, examine the following diagrams.



$\frac{3}{16}$ $\qquad$ $\frac{1}{4}$ $\qquad$ $\frac{3}{8}$

The first diagram is split into 16 rectangles. 3 of the 16 rectangles are shaded to represent the number of students who earned A's. The second diagram is split into 4 rectangles. 1 of the 4 rectangles is shaded to represent the number of students who earned B's. The third diagram is split into 8 rectangles. 3 of the 8 rectangles are shaded to represent the number of students who earned C's.

To find the sum, add the shaded regions. It is difficult to add these regions because the rectangles are all different sizes. Divide each original rectangle into sixteen smaller rectangles to make all the rectangles the same size.



$\frac{3}{16}$ $\qquad$ $\frac{1}{4}=\frac{4}{16}$ $\qquad$ $\frac{3}{8}=\frac{6}{16}$

Now that the rectangles are all the same size, it is easy to see there are a total of thirteen shaded rectangles. Therefore,

$$\frac{3}{16}+\frac{1}{4}+\frac{3}{8}=$$

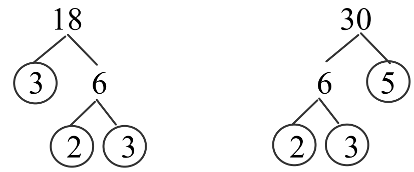
$$\frac{3}{16}+\frac{4}{16}+\frac{6}{16}=\boxed{\frac{13}{16}}$$ 13 out of 16 students passed the class.

This process is called "finding the least common denominator".

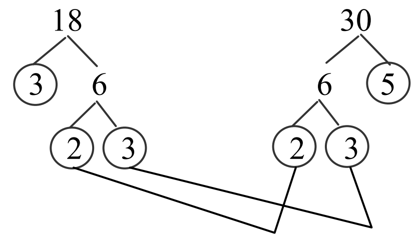
The **least common denominator**, or **LCD**, of two or more fractions, is the smallest number that is a multiple of each denominator.

Example 1 **Rewrite the fractions $\frac{11}{18}$ and $\frac{13}{30}$ with an LCD.**

Make a factor tree for each denominator.



Find the prime factors they have in common. (If you have three or more numbers, your common factors need to appear in at least two of them.)



In this problem, the numbers have a 2 and a 3 in common.

Multiply the common factors, 2 and 3, along with any numbers that are not in common, in this case 3 and 5. So the LCD is:

$$\text{LCD} = 2 \cdot 3 \cdot 3 \cdot 5$$
$$\text{LCD} = 90$$

Once a common denominator is found, rewrite each fraction with that LCD. To do this, find the form of 1 that can be used to multiply the original fraction by to get the denominator of 90. The first fraction has the denominator of 18. What can we multiply 18 by to get 90? To find this number, use division. $90 \div 18 = 5$. The 1 will take the form of $\frac{5}{5}$. Multiply $\frac{11}{18}$ by $\frac{5}{5}$. Follow the same steps for the second fraction. $90 \div 30 = 3$. The 1 will take the form of $\frac{3}{3}$. Multiply $\frac{13}{30}$ by $\frac{3}{3}$.

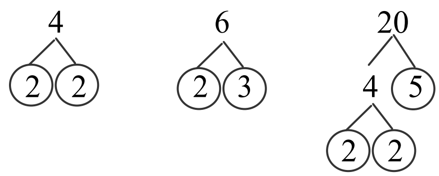
$$\frac{11}{18} \cdot \frac{5}{5} \quad \text{and} \quad \frac{13}{30} \cdot \frac{3}{3}$$

$$\boxed{\frac{55}{90}} \quad \text{and} \quad \boxed{\frac{39}{90}}$$

Now the fractions have an LCD of 90.

Example 2

Rewrite the fractions $\frac{3}{4}, \frac{5}{6},$ and $\frac{1}{20}$ with an LCD.



Prime factor the denominators to find the LCD.

$$\text{LCD} = 2 \cdot 2 \cdot 3 \cdot 5$$
$$\text{LCD} = 60$$

Multiply by a form of 1 to rewrite each fraction with the LCD.

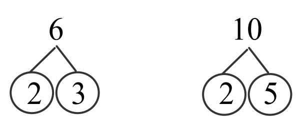
$$\frac{3}{4} \cdot \frac{15}{15} = \frac{45}{60} \quad (60 \div 4 = 15)$$
$$\frac{5}{6} \cdot \frac{10}{10} = \frac{50}{60} \quad (60 \div 6 = 10)$$
$$\frac{1}{20} \cdot \frac{3}{3} = \frac{3}{60} \quad (60 \div 20 = 3)$$

$$\boxed{\frac{45}{60}, \frac{50}{60}, \text{ and } \frac{3}{60}}$$

Now the fractions have an LCD of 60.

Example 3

Rewrite the fractions $\frac{1}{6}$ and $-\frac{7}{10}$ with an LCD.



Prime factor the denominators to find the LCD.

$$\text{LCD} = 2 \cdot 3 \cdot 5$$
$$\text{LCD} = 30$$

$$\frac{1}{6} \cdot \frac{5}{5} = \frac{5}{30} \quad (30 \div 6 = 5)$$
$$-\frac{7}{10} \cdot \frac{3}{3} = -\frac{21}{30} \quad (30 \div 10 = 3)$$

Multiply by a form of 1 to rewrite each fraction with the LCD.

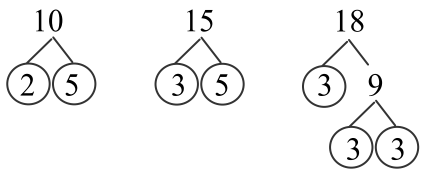
$$\boxed{\frac{5}{30} \text{ and } -\frac{21}{30}}$$

Example 4

Rewrite the fractions $2\frac{9}{10}, \frac{4}{15},$ and $1\frac{5}{18}$ with an LCD.

$$\frac{29}{10}, \frac{4}{15}, \text{ and } \frac{23}{18}$$

First, convert the mixed numbers into improper fractions.



Prime factor the denominators to find the LCD.

$$\text{LCD} = 2 \cdot 3 \cdot 3 \cdot 5$$

$$\text{LCD} = 90$$

Multiply by a form of 1 to rewrite each fraction with the LCD.

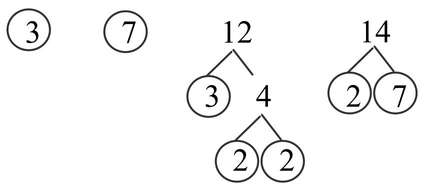
90÷10=9   90÷15=6   90÷18=5

$$\frac{29}{10} \cdot \frac{9}{9} = \frac{261}{90} \qquad \frac{4}{15} \cdot \frac{6}{6} = \frac{24}{90} \qquad \frac{23}{18} \cdot \frac{5}{5} = \frac{115}{90}$$

$$\boxed{\frac{261}{90}, \quad \frac{24}{90}, \quad \text{and} \quad \frac{115}{90}}$$

Example 5

Rewrite the fractions $-\frac{1}{3}, \frac{2}{7}, -\frac{5}{12},$ and $\frac{3}{14}$ with an LCD.



Prime factor the denominators to find the LCD.

$$\text{LCD} = 2 \cdot 2 \cdot 3 \cdot 7$$

$$\text{LCD} = 84$$

Multiply by a form of 1 to rewrite each fraction with the LCD.

84÷3=28   84÷7=12   84÷12=7   84÷14=6

$$-\frac{1}{3} \cdot \frac{28}{28} = -\frac{28}{84} \qquad \frac{2}{7} \cdot \frac{12}{12} = \frac{24}{84} \qquad -\frac{5}{12} \cdot \frac{7}{7} = -\frac{35}{84} \qquad \frac{3}{14} \cdot \frac{6}{6} = \frac{18}{84}$$

$$\boxed{-\frac{28}{84}, \quad \frac{24}{84}, \quad -\frac{35}{84}, \quad \text{and} \quad \frac{18}{84}}$$

**If you are having difficulty finding the LCD, you can find a common denominator, not necessarily the LCD, by multiplying the denominators of each fraction together. If you do this, you will most likely be required to do additional reducing of your final answer.

Now that we know how to rewrite fractions with the least common denominator, let's look at the steps for addition and subtraction of fractions.

> **ADDITION AND SUBTRACTION OF FRACTIONS**
>
> If you have a common denominator,
> 1. Add or subtract the **numerators**, write that sum or difference over the common denominator.
> 2. Reduce if possible.
>
> $$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c} \qquad \frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$$
>
> If you do **<u>not</u>** have a common denominator,
> 1. Write each fraction with a common denominator.
> 2. Add or subtract the **numerators**, write that sum or difference over the common denominator.
> 3. Reduce, if possible.

Example 6

Find the difference. Reduce to lowest terms. $\frac{8}{11}-\frac{5}{11}$

$$\frac{8}{11}-\frac{5}{11} = \frac{8-5}{11}$$

There is a common denominator, so just find the difference.

$$= \boxed{\frac{3}{11}}$$

Example 7

Find the sum. Reduce to lowest terms. $\frac{5}{16}+\frac{1}{16}$

$$\frac{5}{16}+\frac{1}{16} = \frac{5+1}{16}$$

There is a common denominator, so just find the sum.

$$= \frac{\cancel{6}^{3}}{\cancel{16}_{8}}$$

Reduce.

$$= \boxed{\frac{3}{8}}$$

Example 8

Find the sum. Reduce to lowest terms. $1\frac{7}{12}+\frac{5}{18}$

$$1\frac{7}{12}+\frac{5}{18} = \frac{19}{12}+\frac{5}{18}$$

Rewrite the mixed number as an improper fraction.

$$= \frac{19}{12}\cdot\frac{3}{3}+\frac{5}{18}\cdot\frac{2}{2}$$

Find a common denominator. The LCD is 36. Multiply by a form of 1 to rewrite each fraction with this LCD.

$$= \frac{57}{36}+\frac{10}{36}$$

$$= \frac{57+10}{36}$$

Add.

$$= \boxed{\frac{67}{36}}$$

Example 9

Find the sum. Reduce to lowest terms. $-\frac{5}{12}+\frac{-1}{4}$

$$-\frac{5}{12}+\frac{-1}{4} = -\frac{5}{12}+\frac{-1}{4}\cdot\frac{3}{3}$$

Find a common denominator. The LCD is 12. Multiply by a form of 1 to rewrite each fraction with this LCD.

$$= \frac{-5}{12}+\frac{-3}{12}$$

$$= \frac{-5+-3}{12}$$

Add.

$$= \frac{\overset{-2}{\cancel{-8}}}{\underset{3}{\cancel{12}}}$$

Reduce.

$$= \boxed{\frac{-2}{3}}$$

Example 10

Find the difference. Reduce to lowest terms. $2-\frac{3}{4}$

$2-\frac{3}{4} = \frac{2}{1}-\frac{3}{4}$ — *Rewrite the whole number as an improper fraction.*

$= \frac{2}{1}\cdot\frac{4}{4}-\frac{3}{4}$ — *Find a common denominator. The LCD is 4. Multiply by a form of 1 to rewrite the first fraction with this LCD.*

$= \frac{8}{4}-\frac{3}{4}$

$= \frac{8-3}{4}$ — *Subtract.*

$= \boxed{\frac{5}{4}}$

Example 11

Perform the indicated operations. Reduce to lowest terms. $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}$

$\frac{1}{2}+\frac{2}{3}+\frac{3}{4} = \frac{1}{2}\cdot\frac{6}{6}+\frac{2}{3}\cdot\frac{4}{4}+\frac{3}{4}\cdot\frac{3}{3}$ — *Find a common denominator. The LCD is 12. Multiply by a form of 1 to rewrite each fraction with this LCD.*

$= \frac{6}{12}+\frac{8}{12}+\frac{9}{12}$

$= \frac{6+8+9}{12}$ — *Add.*

$= \boxed{\frac{23}{12}}$

Example 12

Perform the indicated operations. Reduce to lowest terms.

$$\frac{7}{12}+\frac{11}{15}-\frac{7}{18}$$

$$\frac{7}{12}+\frac{11}{15}-\frac{7}{18} = \frac{7}{12}\cdot\frac{15}{15}+\frac{11}{15}\cdot\frac{12}{12}-\frac{7}{18}\cdot\frac{10}{10}$$

Find a common denominator. The LCD is 180. Multiply by a form of 1 to rewrite each fraction with this LCD.

$$= \frac{105}{180}+\frac{132}{180}-\frac{70}{180}$$

$$= \frac{105+132-70}{180}$$

Add and subtract in order from left to right.

$$= \boxed{\frac{167}{180}}$$

Example 13

***Application*. Calissa runs $\frac{7}{8}$ of a mile. Tyler runs $\frac{7}{12}$ of a mile. Daisy runs $\frac{1}{4}$ of a mile. What is the sum of the distance they ran?**

To solve this problem, we must add all three fractions together.

$$\frac{7}{8}+\frac{7}{12}+\frac{1}{4}$$

$$\frac{7}{8}+\frac{7}{12}+\frac{1}{4} = \frac{7}{8}\cdot\frac{3}{3}+\frac{7}{12}\cdot\frac{2}{2}+\frac{1}{4}\cdot\frac{6}{6}$$

Find a common denominator.

$$= \frac{21}{24}+\frac{14}{24}+\frac{6}{24}$$

$$= \frac{21+14+6}{24}$$

Add.

$$= \frac{41}{24} \quad or \quad 1\frac{17}{24}$$

The sum of the distance ran is $\frac{41}{24}$ miles or $1\frac{17}{24}$ miles.

3.4 EXERCISES

In 1-8, rewrite fractions with an LCD.

1. $\frac{2}{5}$ *and* $-\frac{7}{15}$

2. $\frac{9}{11}, \frac{5}{6},$ *and* $\frac{2}{3}$

3. $\frac{3}{5}$ *and* $\frac{4}{9}$

4. $\frac{-11}{15}$ *and* $\frac{-5}{12}$

5. $\frac{1}{15}, \frac{3}{20},$ *and* $\frac{5}{24}$

6. $\frac{11}{12}$ *and* $\frac{3}{14}$

7. $\frac{13}{30}, \frac{11}{18},$ *and* $\frac{1}{24}$

8. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4},$ *and* $\frac{1}{5}$

In 9-20, perform the indicated operations. Reduce all answers to lowest terms.

9. $-\frac{7}{10}-\frac{17}{10}$

10. $\frac{2}{3}-\left(-\frac{1}{3}\right)$

11. $\frac{4}{5}+\left(-\frac{5}{6}\right)$

12. $\frac{2}{3}+\frac{1}{4}$

13. $\frac{7}{8}-\frac{5}{7}$

14. $\frac{-4}{15}+\frac{17}{45}$

15. $2\frac{1}{3}+1\frac{2}{9}$

16. $4\frac{1}{5}+\frac{4}{15}$

17. $\frac{5}{6}+\frac{7}{12}+\frac{1}{24}$

18. $10\frac{1}{2}-3\frac{5}{6}$

19. $\left(-\frac{13}{12}\right)-\frac{2}{3}$

20. $\frac{6}{5}-\frac{1}{15}-\frac{1}{3}$

In 21-24, perform the indicated operation. Reduce all answers to lowest terms.

21. What is the sum of $\frac{2}{3}, \frac{4}{5}, \textit{and } \frac{11}{15}$?

22. Find the difference between $\frac{2}{9} \textit{ and } \frac{5}{6}$.

23. Find the sum of $\frac{9}{4} \textit{ and } -\frac{3}{2}$.

24. What is the difference between $\frac{9}{10} \textit{ and } \frac{2}{15}$?

In 25-30, solve the following application problems.

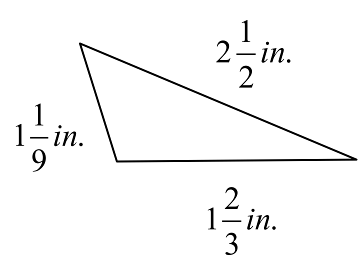
25. A recipe calls for one and one-half cups of sugar for a cake and two-thirds cup of sugar for the frosting. How many cups of sugar does it take to make the entire recipe?

26. During the pie eating contest my dad ate five and one-fifths pies and my mom ate two and one-fourth pies. How many pies did they eat altogether?

27. Cindee is trying to get into shape. She runs one-half of a mile on Monday, two-thirds of a mile on Tuesday, three-fourths of a mile on Wednesday, five-sixth of a mile on Thursday and five-fourths miles on Friday. How far did she run this week?

28. The Ison family spends one-third of their income on rent and one-sixth of their income on food. What fraction of their income is left?

29. The distance around a triangle is called the perimeter. To find the perimeter of a triangle, add the lengths of the sides. What is the perimeter of the following triangle?



30. Alesha bought fifteen cups of flour. She made cookies and used two and one-third cups. Then she made a dinner dish using one and one-half cups. She then made a loaf of bread to go with the dinner. The bread required four and five-sixths cups of flour. How much flour does Alesha have left?

3.5 Order of Operations with Fractions & Complex Fractions

ORDER OF OPERATIONS

Each section of this chapter has focused on a particular mathematical operation with expressions involving fractions. This provided a foundation for simplifying expressions involving fractions on a basic level. Now that we have a foundation, we can face the challenge of dealing with numerous mathematical operations within a single expression. To simplify fraction expressions that involve more than one of these operations, apply the order of operations. **Always remember to reduce your final answer to lowest terms.**

> **ORDER OF OPERATIONS**
>
> 1. Simplify within **parentheses** ( ) and other grouping symbols, such as **brackets** [ ], **braces** { }, or the **fraction bar —**. *(When more than one pair of grouping symbols occur within a problem, work the innermost set of grouping symbols first.)*
> 2. Evaluate **exponents** and/or **roots**.
> 3. **Multiply** and/or **divide** in order from left to right.
> 4. **Add** and/or **subtract** in order from left to right.

Example 1 **Simplify the following expression.** $\frac{3}{8}\cdot\frac{5}{9}+\frac{3}{4}$

$$\frac{3}{8}\cdot\frac{5}{9}+\frac{3}{4} = \frac{\cancel{3}^{1}}{8}\cdot\frac{5}{\cancel{9}_{3}}+\frac{3}{4} = \frac{5}{24}+\frac{3}{4}$$

Multiply first.

$$= \frac{5}{24}+\frac{3}{4}\cdot\frac{6}{6} = \frac{5}{24}+\frac{18}{24}$$

Find a common denominator, then add.

$$= \boxed{\frac{23}{24}}$$

Example 2 **Simplify the following expression.** $\frac{2}{3}+\left(-\frac{1}{4}\right)^2$

$$\frac{2}{3}+\left(-\frac{1}{4}\right)^2 = \frac{2}{3}+\frac{1}{16}$$

Simplify exponents. Note: a negative times a negative is a positive.

$$= \frac{2}{3}\cdot\frac{16}{16}+\frac{1}{16}\cdot\frac{3}{3} = \frac{32}{48}+\frac{3}{48}$$

Find a common denominator then, add.

$$= \boxed{\frac{35}{48}}$$

Example 3

Simplify the following expression. $2\frac{1}{2}+\frac{1}{3}(6+2)$

$$2\frac{1}{2}+\frac{1}{3}(6+2) = \frac{5}{2}+\frac{1}{3}(6+2)$$

Change the mixed number into an improper fraction. Follow the order of operations by simplifying within the parentheses.

$$= \frac{5}{2}+\frac{1}{3}(8)$$

$$= \frac{5}{2}+\frac{1}{3}\left(\frac{8}{1}\right) = \frac{5}{2}+\frac{8}{3}$$

Rewrite 8 as an improper fraction and multiply.

$$= \frac{5}{2}\cdot\frac{3}{3}+\frac{8}{3}\cdot\frac{2}{2} = \frac{15}{6}+\frac{16}{6}$$

Find a common denominator then, add.

$$= \boxed{\frac{31}{6}}$$

Example 4

Simplify the following expression. $2\frac{1}{4}-1\frac{1}{2}\cdot\frac{3}{4}$

$$2\frac{1}{4}-1\frac{1}{2}\cdot\frac{3}{4} = \frac{9}{4}-\frac{3}{2}\cdot\frac{3}{4}$$

Change the mixed numbers into improper fractions. Follow the order of operations and multiply.

$$= \frac{9}{4}-\frac{9}{8} = \frac{9}{4}\cdot\frac{2}{2}-\frac{9}{8} = \frac{18}{8}-\frac{9}{8}$$

Find a common denominator then, subtract.

$$= \boxed{\frac{9}{8}}$$

Example 5

Simplify the following expression. $\sqrt{\frac{16}{25}}\left[\left(\frac{2}{3}\right)^2-\frac{1}{2}\cdot\frac{3}{4}\right]$

$$\sqrt{\frac{16}{25}}\left[\left(\frac{2}{3}\right)^2-\frac{1}{2}\cdot\frac{3}{4}\right] = \sqrt{\frac{16}{25}}\left[\frac{4}{9}-\frac{1}{2}\cdot\frac{3}{4}\right]$$

Follow the order of operations by simplifying within the parentheses.

$$= \sqrt{\frac{16}{25}}\left[\frac{4}{9}-\frac{3}{8}\right]$$

$$= \sqrt{\frac{16}{25}}\left[\frac{4}{9}\cdot\frac{8}{8}-\frac{3}{8}\cdot\frac{9}{9}\right] = \sqrt{\frac{16}{25}}\left[\frac{32}{72}-\frac{27}{72}\right]$$

Find a common denominator then, subtract.

$$= \sqrt{\frac{16}{25}}\left[\frac{5}{72}\right]$$

Simplify the square root by finding the square root of the numerator and the square root of the denominator.

$$= \frac{\overset{1}{\cancel{4}}}{\underset{1}{\cancel{5}}}\left[\frac{\overset{1}{\cancel{5}}}{\underset{18}{\cancel{72}}}\right]$$

Multiply and reduce.

$$= \boxed{\frac{1}{18}}$$

Example 6

Simplify the following expression. $\left(\frac{1}{2}+\frac{2}{3}\right)\left(\frac{1}{7}+\frac{5}{7}\right)$

$$\left(\frac{1}{2}+\frac{2}{3}\right)\left(\frac{1}{7}+\frac{5}{7}\right) = \left(\frac{1}{2}\cdot\frac{3}{3}+\frac{2}{3}\cdot\frac{2}{2}\right)\left(\frac{1}{7}+\frac{5}{7}\right)$$

Follow the order of operations by simplifying within the parentheses.

$$= \left(\frac{3}{6}+\frac{4}{6}\right)\left(\frac{1}{7}+\frac{5}{7}\right)$$

Add.

$$= \left(\frac{\overset{1}{\cancel{7}}}{\underset{1}{\cancel{6}}}\right)\left(\frac{\overset{1}{\cancel{6}}}{\underset{1}{\cancel{7}}}\right)$$

Multiply and reduce.

$$= \boxed{1}$$

COMPLEX FRACTIONS

Complex fractions sound difficult, but they are not as complex as they sound. A **complex fraction** is a fraction that contains one or more fractions in its numerator, denominator, or both. The following are examples of complex fractions:

$$\frac{\frac{2}{5}}{\frac{7}{10}}, \quad \frac{3}{\frac{4}{5}}, \quad \frac{\frac{1}{2}-\frac{1}{3}}{\frac{3}{7}}$$

To simplify a complex fraction start by rewriting the expression as a division problem. If the numerator or denominator contains more than one term, enclose them in a set of parentheses. Follow your order of operations to simplify the expression.

Example 7 **Simplify the following expression.** $\dfrac{\frac{3}{10}}{\frac{1}{2}}$

$$\frac{\frac{3}{10}}{\frac{1}{2}} = \frac{3}{10} \div \frac{1}{2}$$ *Rewrite as a division problem.*

$$= \frac{3}{10} \cdot \frac{2}{1} = \frac{3}{\cancel{10}_{5}} \cdot \frac{\cancel{2}^{1}}{1}$$ *Invert and multiply. Then reduce.*

$$= \boxed{\frac{3}{5}}$$

Example 8 **Simplify the following expression.** $\dfrac{2\frac{1}{2}}{9}$

$$\frac{2\frac{1}{2}}{9} = 2\frac{1}{2} \div 9$$ *Rewrite as a division problem.*

$$= \frac{5}{2} \div \frac{9}{1}$$ *Change the mixed number and the whole number to improper fractions.*

$$= \frac{5}{2} \cdot \frac{1}{9}$$ *Invert and multiply.*

$$= \boxed{\frac{5}{18}}$$

Example 9

Simplify the following expression. $\dfrac{\frac{2}{5}+\frac{1}{6}}{\frac{3}{4}-\frac{7}{8}}$

$$\frac{\frac{2}{5}+\frac{1}{6}}{\frac{3}{4}-\frac{7}{8}} = \left(\frac{2}{5}+\frac{1}{6}\right)\div\left(\frac{3}{4}-\frac{7}{8}\right)$$

Rewrite as a division problem. Use parentheses for both the numerator and denominator.

$$= \left(\frac{2}{5}\cdot\frac{6}{6}+\frac{1}{6}\cdot\frac{5}{5}\right)\div\left(\frac{3}{4}\cdot\frac{2}{2}-\frac{7}{8}\right)$$

Simplify within each group of parentheses. Start by finding an LCD.

$$= \left(\frac{12}{30}+\frac{5}{30}\right)\div\left(\frac{6}{8}-\frac{7}{8}\right)$$

Add and Subtract.

$$= \left(\frac{17}{30}\right)\div\left(-\frac{1}{8}\right)$$

$$= \frac{17}{30}\cdot-\frac{8}{1} = \frac{17}{\cancel{30}_{15}}\cdot-\frac{\cancel{8}^{4}}{1}$$

Invert and multiply. Then reduce.

$$= \boxed{-\frac{68}{15}}$$

Example 10

Simplify the following expression. $\dfrac{5-\frac{3}{4}}{2-\frac{1}{3}}$

$$\frac{5-\frac{3}{4}}{2-\frac{1}{3}} = \left(5-\frac{3}{4}\right)\div\left(2-\frac{1}{3}\right) = \left(\frac{5}{1}-\frac{3}{4}\right)\div\left(\frac{2}{1}-\frac{1}{3}\right)$$

Rewrite as a division problem. Change the whole numbers, 5 and 2, into improper fractions. Use parentheses for both the numerator and denominator.

$$= \left(\frac{5}{1}\cdot\frac{4}{4}-\frac{3}{4}\right)\div\left(\frac{2}{1}\cdot\frac{3}{3}-\frac{1}{3}\right)$$

Simplify within each group of parentheses. Start by finding an LCD.

$$= \left(\frac{20}{4}-\frac{3}{4}\right)\div\left(\frac{6}{3}-\frac{1}{3}\right) = \left(\frac{17}{4}\right)\div\left(\frac{5}{3}\right)$$

Add and Subtract.

$$= \frac{17}{4}\cdot\frac{3}{5} = \boxed{\frac{51}{20}}$$

Invert and multiply.

Example 11

Simplify the following expression.

$$\frac{\left(1\frac{1}{2}\right)^2}{\frac{2}{3}+\frac{2}{9}}$$

$$\frac{\left(1\frac{1}{2}\right)^2}{\frac{2}{3}+\frac{2}{9}} = \left(1\frac{1}{2}\right)^2 \div \left(\frac{2}{3}+\frac{2}{9}\right)$$

Rewrite as a division problem. Use parentheses for the denominator. Change the mixed number into an improper fraction.

$$= \left(\frac{3}{2}\right)^2 \div \left(\frac{2}{3}\cdot\frac{3}{3}+\frac{2}{9}\right)$$

Simplify within the parentheses. Start by finding an LCD.

$$= \left(\frac{3}{2}\right)^2 \div \left(\frac{6}{9}+\frac{2}{9}\right)$$

Add.

$$= \left(\frac{3}{2}\right)^2 \div \left(\frac{8}{9}\right)$$

Simplify the exponent.

$$= \left(\frac{9}{4}\right) \div \left(\frac{8}{9}\right)$$

$$= \frac{9}{4}\cdot\frac{9}{8}$$

Invert and multiply.

$$= \boxed{\frac{81}{32}}$$

3.5 EXERCISES

In 1-10, simplify the following expressions. Reduce all answers to lowest terms.

1. $\sqrt{\frac{1}{9}}+\left(\frac{2}{3}\right)^2$

2. $\frac{2}{3}+4\left(\frac{1}{2}+\frac{3}{2}\right)$

3. $\left(\frac{2}{3}\right)^2\left(\frac{3}{4}\right)-\frac{2}{5}$

4. $\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)^2+\frac{1}{8}$

5. $\left(\frac{2}{3}+\frac{2}{9}\right)\left(\frac{1}{2}-\frac{2}{5}\right)$

6. $\frac{2}{3}\left(\frac{3}{4}+1\right)+\left(\frac{1}{3}\right)^2$

7. $\frac{5}{9}\div\left(-\frac{35}{39}\right)\left(1\frac{1}{13}\right)$

8. $\frac{4}{7}\div\left(-\frac{2}{7}\right)\left(\frac{-3}{8}\right)$

9. $\sqrt{\frac{9}{16}}\left[\frac{1}{2}-\left(\frac{3}{4}\right)^2\right]$

10. $1\frac{1}{6}\div\frac{1}{2}+1\frac{1}{5}-\frac{2}{3}$

In 11-20, simplify. Reduce all answers to lowest terms.

11. $\dfrac{\frac{3}{4}}{\frac{7}{8}}$

12. $\dfrac{\frac{1}{3}}{\frac{5}{6}}$

13. $\dfrac{\frac{1}{8}+\frac{3}{4}}{\frac{1}{2}-\frac{1}{3}}$

14. $\dfrac{\frac{2}{3}+\frac{3}{2}}{\frac{1}{3}-\frac{5}{6}}$

15. $\dfrac{2\frac{1}{3}}{9}$

16. $\dfrac{\frac{3}{7}+5}{6-\frac{8}{14}}$

17. $\dfrac{\left(\frac{5}{2}\right)^2}{2-\frac{1}{2}}$

18. $\dfrac{7}{3\frac{5}{6}}$

19. $\dfrac{2\frac{1}{3}+\frac{2}{9}}{\left(\frac{2}{3}\right)^2}$

20. $\dfrac{7-\frac{1}{2}}{3-\frac{1}{8}}$

CHAPTER 3 REVIEW EXERCISES

In 1-6, identify the following as a proper fraction, improper fraction, or mixed number.

1. $3\frac{1}{7}$

2. $-\frac{11}{3}$

3. 4

4. $\frac{13}{13}$

5. $\frac{2}{3}$

6. $-12\frac{3}{8}$

In7-10, change the following improper fractions into mixed numbers using division.

7. $\frac{7}{4}$

8. $\frac{100}{3}$

9. $-\frac{19}{13}$

10. $\frac{28}{5}$

In 11-14, change the following mixed numbers into improper fractions.

11. $11\frac{1}{3}$

12. $-4\frac{2}{5}$

13. $2\frac{11}{13}$

14. $-7\frac{4}{9}$

In 15-20, reduce or simplify the following fractions to lowest terms.

15. $\frac{120}{200}$

16. $-\frac{24a^2b}{18ab^5}$

17. $-\frac{192}{264}$

18. $\frac{125x^4y^5z}{15x^0y^6z^2}$

19. $\frac{a^3bc^4d^0}{a^2b^3d}$

20. $\frac{160r^2st}{252rs^2t^4}$

In 21-38, perform the indicated operations. Remember to reduce your answer to lowest terms.

21. $2\frac{3}{4}\cdot\frac{8}{9}$

22. $\frac{a^2b}{c}\cdot\frac{c^3}{ab^2}$

23. Find the product of $\frac{18x^2}{y}$ and $\frac{y^2}{9x}$

24. $\dfrac{\frac{6}{7}}{\frac{3}{14}}$

25. $-\frac{5}{6}\div\frac{5}{6}$

26. Divide $\frac{1}{84}$ by $\frac{3}{49}$

27. $\frac{11}{20}\div\frac{3}{5x}\div\frac{40x^2}{12x}$

28. $\frac{1}{8}+\frac{5}{8}$

29. $\frac{7}{12}+\frac{3}{4}$

30. Find the sum of $\frac{1}{3}$ and $1\frac{5}{18}$

31. $3\frac{1}{4}-1\frac{1}{2}$

32. $-2\frac{3}{4}-\frac{7}{12}$

33. Find the difference between $\frac{11}{27}$ and $\frac{1}{18}$

34. $\sqrt{\frac{9}{25}}\div\left(-\frac{3}{25}\right)\left(-\frac{3}{11}\right)$

35. $\frac{1}{3}+\frac{2}{9}\left(3\frac{1}{2}+3\frac{1}{2}\right)$

36. $\left(\frac{2}{3}\right)^2\left(\frac{3}{4}\right)-\frac{1}{5}$

37. $\dfrac{\frac{3}{4}+\frac{1}{3}}{\frac{2}{3}+\frac{1}{6}}$

38. $\dfrac{\frac{2}{7}+5}{6-\frac{5}{14}}$